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Editors

Dr. Aliaa Burqan, Dr. Osama Ababneh and Dr. Shawkat Alkhazaleh

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FOREWORD

Aliaa Burqan
Conference Chairman
Zarqa University

I am happy to introduce to you the 6th International Arab Conference on Mathematics and Computations, IACMC 2019. This conference is among a series of international conferences held and sponsored by Zarqa University. The purpose of all IACMC's is to bring together researchers and professionals in all fields of Mathematical Sciences to meet, discuss, to share and explore ideas that improve their research. On the other hand, these conferences will also provide a good opportunity to encourage young researchers, students and all those who are desirous of working in the field of Mathematics to in tract with each other and to explore possibilities for future collaborative work.

This book contains the short papers of IACMC 2019 which is held in Zarqa University on April 24-26, 2019. This sixth edition contains a large number of research topics and applications in both pure and applied mathematics in addition to the field of statistics which are the topics included in the scope of IACMC's. Furthermore, the program is enriched by several keynote lectures delivered by well-known experts in their areas of Mathematics.

IACMC 2019 received 120 abstract submissions from 20 countries. The accepted full-papers went through an evaluation method: each paper was reviewed by two reviewers from the IACMC Scientific Committee; one of them is an international known expert. Authors of some selected papers, based on the reviewer's evaluations and on the oral presentations, are invited to submit extended versions of their papers for a book which will be published by Springer.

The program for this conference required the dedicated effort of many people. Firstly, we must thank the sponsors of IACMC 2019: Zarqa University and The Scientific Research Support Fund. Secondly, we thank the invited speakers for their invaluable contributions and the authors, whose research efforts are herewith recorded. We also give our thanks to the reviewers for their diligent and professional reviewing. Last but not least, a special word of thanks is due to those who spent much of their time to make the success of this conference: to all members of the Local and Organizing Committees for their super job.

We look forward to welcoming and sharing this conference with you. Wishing you all an exciting conference and an unforgettable stay in Jordan and hoping to meet you again for the 7th IACMC.

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CONTENTS

NEW TYPE CONTRACTIVE CONDITIONS FOR KANNAN AND CHATTERJEA FIXED POINT THEOREMS IN B- METRIC SPACES

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ABSTRACT

Centrale to the entire discipline of mathematics is the concept of space that has heatedly received considerable attention in the last few decades. B-metric spaces, in particular is a major area of interesting within types of spaces. In essence, the present study seeks at extreme the one proceding a clear insight of the concept b metric and establishing its main structure. At the other extreme, it attempts to introduce a new class principle contraction to prove kannan and chatterjea fixed point theorems. We also give in the end of paper some examples to illustrate the given results

Keywords: B-metric space; Fixed point; Cauchy sequence.

1. INTRODUCTION

The metric spaces are the well known space and very important tool for all branches of mathematics. The first important result in the theory of fixed point about contractive mapping is Banach theorem.
A mapping $T : X \to X$, where (X,d) is a metric space, wich is a contraction if there exists k in $[0,1)$ such that for all x,y in X,

$$d(f(x),f(y)) \leq d(x,y).$$

Additionally, there are numerous generalization of usual metric spaces.We refer the readers to *[1],[6],[8,9,10]*. One of them is b-metric space, b-metric spaces are one of the among spaces which generalize the classical metric.Czerwik [8] is the first presented a generalization of banach fixed point theorem in b-metric spaces.
This recherches introduced some classes of contractive principle and proved some theorems in b-metric spaces by imposing some additional conditions.
In the present paper, we extend and prove the Kannan's and Chatterjea's theorem in b-metric spaces with new contractive principle. At the end of paper, we introduce an example to illustrate our results.

2. PRELIMINARIES

B-metric spaces could be defined by disparate scholars as follow :

Definition 2.1. *[2]* Let X be a nonempty set and $d: X \times X \to [0,+\infty)$. A function d is called a b-metric with constant $s \geq 1$ if

$b(0)$ $d(x, y) = 0$ if and only if $x = y$;

$b(1)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$b(2)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.

In this case, the pair (X, d) is called a b-metric space.

Obviously, a b-metric space with base $s = 1$ is a metric space. Moreover, we can consider every metric space as a b-metric space but contrary is not necessary true. A well-known example of b-metric spaces are given below

Example 2.2 *[4]* Let $X=\{0,1,2\}$ and $d(0,2) = d(2, 0) = m>2$,
$d(0,1) = d(1, 2) = d(1,0)=d(2,1)=1$,
and $d(0, 0) = d(1, 1) = d(2, 2) = 0$. Then $d(x,y) \leq (m/2)(d(x,z)+d(z,y))$ for all x,y,z in X

Definition 2.3.*[3]* Let $\{x_n\}$ be a sequence in a *b*-metric space (X, d).
(1) A sequence $\{x_n\}$ is called convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ when $n \to +\infty$.
(2) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, when $n,m \to +\infty$.

Definition 2.4.*[3]* The *b*-metric space is complete if every Cauchy sequence convergent.

Lemma 2.5. Let $\{x_n\}$ be a sequence in a b-metric type space (X,d) such that

$$d(x_n,x_{n+1}) \leq \lambda d(x_n,x_{n-1}),$$

for some $\lambda, 0<\lambda<1/s$, and each $n= 1,2,\ldots$. Then $\{x_n\}$ is a Cauchy sequence in (X,d).

3. MAIN RESULT

Throughout this section, we afford two fixed point theorems in *b*-metric spaces. The first one theorem is about Kannan's contraction and the second one is about Chatterjea contraction in *b*-metric spaces.

Theorem 3.1. Let (X, d) be a complete *b*-metric space with constant $s \geq 1$.
If $a>0,\ b\geq 0,\ (2a+b)<1$ and

$$d(Tx,Ty) \leq a(d(x,Tx) + d(y,Ty)) + bd(x,y) \qquad (1)$$

for all x, y in X, then there is a unique fixed point on T

Proof. Let x in X and x be a sequence in X defined as following

$$Tx_n=x_{n+1}, \quad n=0,1,2\ldots$$

By using *(1),*

$$d(x_n,x_{n+1}) \leq a(d(x_n,x_{n+1}) + d(x_{n-1},x_n)) + bd(x_n, x_{n-1})$$
$$\leq ad(x_n,x_{n+1}) + a\, d(x_{n-1},x_n) + bd(x_n, x_{n-1}).$$

we get

$$d(x_n,x_{n+1}) \leq (a+b)/(1-a)\ d(x_{n-1},x_n).$$

By repeating this procedure, we get

$$d(x_n,x_{n+1}) \leq [(a+b)/(1-a)]^n\ d(x_1,x_0).$$

Using $(a+b/1-a)<1$, we get, T is a contraction mapping.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$,

$$d(x_n,x_m) \leq c^n d(x_1,x_0)$$

for each $n =0,1,2,3,\ldots$, and

$$0\leq c=(a+b/1-a) <1$$

Then the sequence $\{x_n\}$ is a Cauchy sequence in X. In view of completeness of X; we consider that $\{x_n\}$ convergent to x^* in X.

4. UNIQUENESS OF FIXED POINT:

Finally, we have to show that the fixed point is unique. Assume that is another fixed point of $T\,x'= x'$. This case is a contradiction with condition (1). So the fixed point is unique. This completes the proof □

Our next theorem about Chatterjea type fixed point theorem in b-metric spaces with new contractive condition.

Theorem 3.2. Let *(X, d)* be a complete b-metric space with constant $s \geq 1$.
If $a>0, b\geq 0, 2sa+b<1$ and

$$d(Tx,Ty) \leq a(d(y,Tx) + d(x,Ty)) + bd(x,y) \qquad (2)$$

for all x, y *in* X, then there is a unique fixed point on S

Proof. Let x in X and $\{x_n\}$ be a sequence in X defined as following

$$Tx_n=x_{n+1}, \quad n=0,1,2...$$

By using *(2),*

$$d(x_n,x_{n+1}) \leq a(d(x_{n-1},Sx_n) + d(x_n,Sx_{n-1})) + bd(x_n, x_{n-1})$$
$$\leq ad(x_{n-1},x_{n+1}) + bd(x_n, x_{n-1})$$
$$\leq asd(x_{n-1},x_n) + asd(x_{n+1},x_n) + bd(x_n, x_{n-1}),$$

This implies that

$$d(x_n,x_{n+1}) \leq (as+b)/(1-sa)\ d(x_{n-1},x_n).$$

So

$$d(x_n,x_{n+1}) \leq [(as+b)/(1-sa)]^n\ d(x_1,x_0).$$

By *condition* $2sa+b<1$. Thus T is a contraction mapping.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$,

$$d(x_n,x_m) \leq r^n d(x_1,x_0)$$

for each $n =0,1,2,3,...$, and

$$0\leq r=(as+b/1-as)<1$$

Then the sequence $\{x_n\}$ is a Cauchy sequence in X by completeness of X; we consider that $\{x_n\}$ convergent to x^* in X.

5. UNIQUENESS OF FIXED POINT:

The proof of uniqueness is similar to the proof of uniqueness in theorem 3.1.□

Remarks 3.3 If we take $s= 1, b=0$ and $S=f$,Theorem 3.1 reduce to Kannan theorem *[7]* and if we take $s= 1, b=0$ and $S=f$, Theorem 3.2 would be the Chatterjea theorem *[5]*.

Example 3.4 Let $X=\{0,1,2\}$ and $d:X.X\rightarrow [0,+\infty[$ be defined as follows:
$d(0,0)=d(1,1)=d(2,2)=0$, $d(0,1)=d(1,0)=d(0,2)=d(2,0)=2/7$, $d(1,2)=d(2,1)=5/7$. It is easy to check that *(X,d)* is a *b*-metric space with $s=4/3$ and it is not a metric space (usual).
Define $T:X\rightarrow X$ by $T0=0, T1=2, T2=0$. If we take $a=1/5$ and $b=1/2$ in theorem 3.1, thus the inequality *(1)* holds for all *x,y in X.*

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COMPARING THE EFFICIENCY OF DIFFERENT STARTIFIED SAMPLING METHODS FOR ESTIMATING THE POPULATION MEAN

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ABSTRACT

Many methods related to stratified ranked set sampling are suggested for estimating the population mean. Some of these methods are stratified quartile ranked set sample (SQRSS), stratified percentile ranked set sample (SPRSS), stratified median ranked set sample (SMRSS) and stratified extreme ranked set sample (SERSS). These estimators are compared to stratified simple random sample (SSRS) and stratified ranked set sample (SRSS). It is found that all estimators are unbiased estimators of the population mean and they are more efficient than their counterparts using SSRS and SRSS. A simulation study is considered to compare the efficiency of the above estimators.

Keywords: Ranked set sampling;Stratified;Quartile;Median; Percentile; Extreme; Efficiency.

1. INTRODUCTION

McIntyre (1952), considered the mean of n units based on a ranked set sampling (RSS) to estimate the population mean. Takahasi and Wakimoto (1968) provided the mathematical theory for RSS. Dell and Clutter (1972) showed that the mean of the RSS is an unbiased estimator of the population mean, whether or not there are errors in ranking. Muttlak (1997) suggested median ranked set sampling (MRSS) to estimate the population mean. Muttlak (2003) considered quartile ranked set sampling (QRSS) to estimate the population mean,he showed that QRSS reduces the errors in ranking when compared to RSS. Muttlak (2003b) suggested percentile ranked set sampling (PRSS) to estimate the population mean and he showed using PRSS procedure will reduce the errors in ranking comparing to RSS since we only select and measure the pth or the qth percentile of the sample. K. Ibrahim, Al-Omari and Syam (2010) estimated the population mean using SMRSS, then in (2012) they estimated the population mean using SQRSS and SERSS.
The aim of this paper is to compare some suggested estimators for the population mean as stratified quartile ranked set sample (SQRSS), stratified percentile ranked set sample (SPRSS), stratified median ranked set sample (SMRSS) and stratified extreme ranked set sample (SERSS). These estimators are more efficient than those obtained based on stratified simple random sample (SSRS) and stratified ranked set sampling (SRSS).

2. SAMPLING METHODS

2.1. *Ranked set sampling*

McIntyre (1952) first suggested the ranked set sampling (RSS) method. The RSS involves selection of n random samples of size n units each from the population and ranking of the units in each sample with respect to the variable of interest. An actual measurement is taken for the unit with the smallest rank from the first sample. From the second sample, an actual measurement is taken for the unit with the second smallest rank, and the procedure is continued until the unit with the largest rank from the nth sample is chosen for actual measurement.

2.2. *MEDIAN RANKED SET SAMPLING*

The MRSS procedure as proposed by Muttlak (1997) depends on selecting n random samples of size *n* units from the population and ranking the units within each sample with respect to a variable of interest. If the sample size *n* is odd then from each sample select for the measurement the $\left(\frac{(n+1)}{2}\right)th$ smallest rank, which means the median of the sample. If the sample size *n* is even then select for the measurement from the first $\frac{n}{2}$ samples the $\left(\frac{n}{2}\right)th$ smallest rank and from the second $\frac{n}{2}$ samples the $\left(\frac{n}{2}+1\right)th$ smallest rank.

2.3. ***Percentile and quartile ranked set sampling***

The PRSS procedure proposed by Muttlak (2003b) depend on selecting n random samples each of size *n* units from the population and rank each sample with respect to a variable of interest. If the sample size *n* is even, then select for measurement from the first $n/2$ samples the $p(n+1)$ th smallest ranked unit and from the second $n/2$ samples the $q(n+1)$ th smallest ranked unit where $0 \le p \le 1$ and $p+q=1$. If the sample size n is odd, then select for measurement from the first $(n-1)/2$ samples the $p(n+1)$ th smallest ranked unit and from the last $(n-1)/2$ samples the $q(n+1)$ th smallest ranked unit, and the median from the middle sample. Quartile ranked set sampling is similar to percentile ranked set sampling but instead of $P(n+1)$ we select q1 and instead of $q(n+1)$ we select q3.

2.4. ***Extreme ranked set sampling***

The ERSS procedure depend on selecting n random samples each of size m units from the population and rank each sample with respect to a variable of interest. If the number of samples *n* is even, then select for measurement from the first $\frac{n}{2}$ samples the smallest rank unit (minimum) and from the second $\frac{n}{2}$ samples the largest rank unit (maximum). If the number of the samples *n* is odd, then select for measurement from the first $\frac{n-1}{2}$ samples the smallest rank unit (minimum) and from the last $\frac{n-1}{2}$ samples the largest rank unit (maximum), and the median from the middle sample.

1.5. ***Stratified sampling***

In stratified sampling the population of N units is first divided into L subpopulations, which are consist of, say, $N_1, N_2, \cdots, N_L$ units. The subpopulations are called strata. To obtain the full benefit from stratification, the size of the hth subpopulation, denoted as N_h where $h=1,2,...,L$, must be known. Once the strata have been determined, samples are drawn independently from the respective strata, producing sample sizes denoted by $n_1, n_2, ..., n_L$, and the total sample size is $n=\sum_{h=1}^{L} n_h$. If a simple random sample is taken from each stratum, the whole procedure is described as stratified simple random sampling (SSRS).

If the ranked set sampling is conducted for each stratum, the whole procedure may be called as stratified ranked set sampling (SRSS). Same for SQRSS, SMRSS, SPRSS and SERSS.

Example 1:

Suppose we have two strata, i.e., $L=2$ and $h=1,2$. Assume that from the first stratum, we draw six samples, each of size 6, and from the second stratum, we draw eight samples each of size 8 as the following:

Stratum 1: Six samples are obtained and ranked as follows:

$X_{11(1)}, X_{11(2)}, X_{11(3)}, X_{11(4)}, X_{11(5)}, X_{11(6)}$ $X_{21(1)}, X_{21(2)}, X_{21(3)}, X_{21(4)}, X_{21(5)}, X_{21(6)}$

$X_{31(1)}, X_{31(2)}, X_{31(3)}, X_{31(4)}, X_{31(5)}, X_{31(6)}$ $X_{41(1)}, X_{41(2)}, X_{41(3)}, X_{41(4)}, X_{41(5)}, X_{41(6)}$

$X_{51(1)}, X_{51(2)}, X_{51(3)}, X_{51(4)}, X_{51(5)}, X_{51(6)}$ $X_{61(1)}, X_{61(2)}, X_{61(3)}, X_{61(4)}, X_{61(5)}, X_{61(6)}$

For the first stratum, h=1,

The chosen elements using SQRSS are: $X_{11(2)}, X_{21(2)}, X_{31(2)}, X_{41(5)}, X_{51(5)}, X_{61(5)}$

The chosen elements using SMRSS are: $X_{11(3)}, X_{21(3)}, X_{31(3)}, X_{41(4)}, X_{51(4)}, X_{61(4)}$

The chosen elements using SPRSS are (Assuming p=40% and q=60%)

$X_{11(2)}, X_{21(2)}, X_{31(2)}, X_{41(4)}, X_{51(4)}, X_{61(4)}$

The chosen elements using SERSS are: $X_{11(1)}, X_{21(1)}, X_{31(1)}, X_{41(6)}, X_{51(6)}, X_{61(6)}$

Same procedure in stratum 2 witheight samples, each of eight units:

Therefore, SQRSS units consist of

$X_{11(2)}, X_{21(2)}, X_{31(2)}, X_{41(5)}, X_{51(5)}, X_{61(5)}, X_{12(2)}, X_{22(2)}, X_{32(2)}, X_{42(2)}, X_{52(6)}, X_{62(6)}, X_{72(6)}\ X_{82(6)}$.

SMRSS units consist of

$X_{11(3)}, X_{21(3)}, X_{31(3)}, X_{41(4)}, X_{51(4)}, X_{61(4)}, X_{12(4)}, X_{22(4)}, X_{32(4)}, X_{42(4)}, X_{52(5)}, X_{62(5)}, X_{72(5)}, X_{82(5)}$

SPRSS units consist of,

$X_{11(2)}, X_{21(2)}, X_{31(2)}, X_{41(4)}, X_{51(4)}, X_{61(4)}, X_{12(3)}, X_{22(3)}, X_{32(3)}, X_{42(3)}, X_{52(5)}, X_{62(5)}, X_{72(5)}\ X_{82(5)}$.

In addition, SERSS consist of $X_{11(1)}, X_{21(1)}, X_{31(1)}, X_{41(6)}, X_{51(6)}, X_{61(6)}, X_{12(1)}, X_{22(1)}, X_{32(1)}, X_{42(1)}, X_{52(8)}, X_{62(8)}, X_{72(8)}, X_{82(8)}$

3. ESTIMATION OF THE POPULATION MEAN

In the case of stratified quartile ranked set sampling (SQRSS), the estimator of the population meanwhen n_h is even and odd are defined as in (1) and (2)

$$\overline{X}_{sqrss1} = \sum_{h=1}^{L} \frac{W_h}{n_h} \left(\sum_{i=1}^{\frac{n_h}{2}} X_{ih(q_1(n_h+1))} + \sum_{i=\frac{n_h+2}{2}}^{n_h} X_{ih(q_3(n_h+1))} \right) \quad (1)$$

Where $W_h = \dfrac{N_h}{N}$, N_h is the stratum size and N is the total population size.

$$\overline{X}_{sqrss2} = \sum_{h=1}^{L} \frac{W_h}{n_h} \left(\sum_{i=1}^{\frac{n_h-1}{2}} X_{ih(q_1(n_h+1))} + \sum_{i=\frac{n_h+3}{2}}^{n_h} X_{ih(q_3(n_h+1))} + X_{(\frac{n_h+1}{2})h(\frac{n_h+1}{2})} \right), \quad (2)$$

The variances of SQRSS1 and SQRSS2are given by

$$\sum_{h=1}^{L} \frac{W_h^2}{n_h^2} \left(\sum_{i=1}^{\frac{n_h}{2}} \sigma^2_{h(i:q_1)} + \sum_{i=\frac{n_h+2}{2}}^{n_h} \sigma^2_{h(i:q_3)} \right) \text{and} \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} \left(\sum_{i=1}^{\frac{n_h-1}{2}} \sigma^2_{h(i:q_1)} + \sum_{i=\frac{n_h+3}{2}}^{n_h} \sigma^2_{h(i:q_3)} + \sigma^2_{h(\frac{n_h+1}{2}:q_2)} \right) \quad (3)$$

In the case of stratified median ranked set sampling (SMRSS), the estimator of the population meanwhen n_h is odd and even are given by

$$\overline{X}_{SMRSS1}=\sum_{h=1}^{L}\frac{W_h}{n_h}\left(\sum_{i=1}^{n_h}X_{ih((n_h+1)/2)}\right),\overline{X}_{SMRSS2}=\sum_{h=1}^{L}\frac{W_h}{n_h}\left(\sum_{i=1}^{\frac{n_h}{2}}X_{ih(n_h/2)}+\sum_{i=\frac{n_h}{2}+1}^{n_h}X_{ih((n_h+2)/2)}\right) \quad (4)$$

The variance of SMRSS1 and SMRSS2are given by

$$Var(\overline{X}_{SMRSS1})=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^2_{h(i:\frac{n_h+1}{2})} \quad ,Var(\overline{X}_{SMRSS2})=\sum_{h=1}^{L}\frac{W_h^2}{n_h^2}\left(\sum_{i=1}^{\frac{n_h}{2}}\sigma^2_{h(i:\frac{n_h}{2})}+\sum_{i=\frac{n_h}{2}+1}^{n_h}\sigma^2_{h(i:\frac{n_h}{2}+1)}\right) \quad (5)$$

In the case of stratified percentile ranked set sampling (SPRSS), the estimator of the population meanwhen n_h is even and oddare defined as

$$\overline{X}_{sprss1}=\sum_{h=1}^{L}\frac{W_h}{n_h}\left(\sum_{i=1}^{\frac{n_h}{2}}X_{ih(p(n_h+1))}+\sum_{i=\frac{n_h}{2}+1}^{n}X_{ih(q(n_h+1))}\right),$$

$$\overline{X}_{sprss2}=\sum_{h=1}^{L}\frac{W_h}{n_h}\left(\sum_{i=1}^{\frac{n_h-1}{2}}X_{ih(p(n_h+1))}+\sum_{i=\frac{n_h-1}{2}+2}^{n_h}X_{ih(q(n_h+1))}+X_{ih(\frac{n_h+1}{2})}\right) \quad (6)$$

The variance of SPRSS1 and SPRSS2are given by

$$Var(\overline{X}_{sprss1})=\sum_{h=1}^{L}\frac{W_h^2}{n_h^2}\left(\sum_{i=1}^{\frac{n_h}{2}}\sigma^2_{h(i:p)}+\sum_{i=\frac{n_h}{2}+1}^{n}\sigma^2_{h(i:q)}\right),$$

$$Var(\overline{X}_{sprss2})=\sum_{h=1}^{L}\frac{W_h^2}{n_h^2}\left(\sum_{i=1}^{\frac{n_h-1}{2}}\sigma^2_{h(i:p)}+\sum_{i=\frac{n_h-1}{2}+2}^{n_h}\sigma^2_{h(i:q)}+\sigma^2_{h(i:q_2)}\right) \quad (7)$$

In the case of stratified extreme ranked set sampling (SERSS), the estimator of the population meanwhen n_h is even and oddare defined as

$$\overline{X}_{SERSS1}=\sum_{h=1}^{L}\frac{W_h}{n_h}\left(\sum_{i=1}^{\frac{n_h}{2}}X_{hi(1)}+\sum_{i=\frac{n_h}{2}+1}^{n_h}X_{hi(m)}\right), \quad \overline{X}_{SERSS2}=\sum_{h=1}^{L}\frac{W_h}{n_h}\left(\sum_{i=1}^{\frac{n_h-1}{2}}X_{hi(1)}+\sum_{i=\frac{n_h+3}{2}}^{n_h}X_{hi(m)}+X_{hi(\frac{m+1}{2})}\right) \quad (8)$$

The variance of SERSS1 and SERSS2are given by

$$=\sum_{h=1}^{L}\frac{W_h^2}{n_h^2}\left(\sum_{i=1}^{\frac{n_h}{2}}\sigma^2_{hi(1)}+\sum_{i=\frac{n_h}{2}+1}^{n_h}\sigma^2_{hi(m)}\right), \quad =\sum_{h=1}^{L}\frac{W_h^2}{n_h^2}\left(\sum_{i=1}^{\frac{n_h-1}{2}}\sigma^2_{hi(1)}+\sum_{i=\frac{n_h+3}{2}}^{n_h}\sigma^2_{hi(m)}+\sigma^2_{h\frac{n_h+1}{2}(\frac{m+1}{2})}\right) \quad (9)$$

Lemma 3.1. *If the distribution is symmetric about* μ*, then*

(a) $\overline{X}_{SQRSS}$ *is an unbiased estimator of the population mean.*

(b) $\overline{X}_{SMRSS}$ *is an unbiased estimator of the population mean.*

(c) $\overline{}_{SPRSS}$ *is an unbiased estimator of the population mean.*

(d) $\overline{X}_{SERSS}$ *is an unbiased estimator of the population mean.*

Lemma 3.2. *If the distribution is symmetricabout* μ*, then*

(a) $Var\left(\overline{X}_{SQRSS1}\right)<Var\left(\overline{X}_{SSRS}\right)$ *and* $Var\left(\overline{X}_{SQRSS2}\right)<Var\left(\overline{X}_{SSRS}\right)$.

(b) $Var\left(\overline{X}_{SMRSS1}\right)<Var\left(\overline{X}_{SSRS}\right)$ *and* $Var\left(\overline{X}_{SMRSS2}\right)<Var\left(\overline{X}_{SSRS}\right)$.

(c) $Var\left(\overline{X}_{SPRSS1}\right)<Var\left(\overline{X}_{SSRS}\right)$ *and* $Var\left(\overline{X}_{SPRSS2}\right)<Var\left(\overline{X}_{SSRS}\right)$.

(d) $Var(\bar{X}_{SERSS1}) < Var(\bar{X}_{SSRS})$ and $Var(\bar{X}_{SERSS2}) < Var(\bar{X}_{SSRS})$

4. SIMULATION STUDY

In this section, a simulation study is conducted to investigate the performance of SQRSS, SMRSS, SPRSS and SERSS for estimating the population mean. Symmetric and asymmetric distributions have been considered for samples of sizes $n=7,14,18$. assuming that the population is partitioned into two or three strata. The simulation was performed for the SRSS and SSRS data sets from different distributions symmetric and asymmetric. The symmetric distributions are uniform and normal, and the asymmetric distributions are geometric and beta. In case of symmetric distributions, the efficiency of estimator T relative to SSRS and SRSS respectively is given by

$$eff(\bar{X}_T, \bar{X}_{SSRS}) = \frac{Var(\bar{X}_{SSRS})}{Var(\bar{X}_T)} \text{ and } eff(\bar{X}_T, \bar{X}_{SRSS}) = \frac{Var(\bar{X}_{SRSS})}{Var(\bar{X}_T)} \quad (10)$$

The values of the relative efficiency found under different distributional assumptions are provided in Table 1.

Table 1: The efficiency of SQRSS, SMRSS, SPRSS and SERSS relative to SRSS and SSRS for $n=7$ and samples sizes $n_1 = 4$ and $n_2 = 3$

Distribution		$\bar{X}_{SPRSS}$ 20%	$\bar{X}_{SQRSS}$	$\bar{X}_{SPRSS}$ 30%	$\bar{X}_{SPRSS}$ 40%	$\bar{X}_{SMRSS}$	$\bar{X}_{SERSS}$
Uniform (0,1)	$\bar{X}_{SRSS}$	1.3440	1.4044	2.1044	2.1137	2.1954	1.2032
	$\bar{X}_{SSRS}$	1.8680	1.9680	2.0097	2.2431	2.3908	1.7873
Normal (0,1)	$\bar{X}_{SRSS}$	1.9521	2.2923	2.3041	2.9172	3.2764	1.8941
	$\bar{X}_{SSRS}$	1.2206	1.3206	1.9804	3.3480	3.7571	1.2007
Geometric (0.5)	$\bar{X}_{SRSS}$	2.6179	3.1237	3.0745	3.0875	3.1237	2.4573
	$\bar{X}_{SSRS}$	2.5990	3.0990	3.0711	3.0725	3.0990	2.3682
Beta (5,2)	$\bar{X}_{SRSS}$	1.1394	1.2593	1.9593	2.5636	2.6154	1.1039
	$\bar{X}_{SSRS}$	1.0606	1.3704	2.1604	2.6636	2.8462	1.0074

5. RESULTS AND DISCUSSION

(1) The suggested estimators SQRSS, SMRSS, SPRSS and SERSS are more efficient than SRSS and SSRS based on the same number of measured units.

(2) When the performance of the suggested estimators are compared, the efficiency of the suggested estimators is found to be more superior when the underlying distributions are symmetric as compared to asymmetric.

(3) The relative efficiency of SQRSS, SMRSS, SPRSS and SERSS estimators to those estimators based on SSRS and SSRS are increasing as the sample size increases.

(4) The relative efficiency of SQRSS, SMRSS, SPRSS and SERSS estimators to those estimators based on SRSS and SSRS are increasing as the percentile increases.

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INTEGRO-DIFFERENTIAL EQUATION METHOD FOR DETERMINATION THE SHAPE OF TWO DIMENSIONAL JET FLOWS IN A SEMI INFINITE TUBE

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ABSTRACT

In this work, we studied mathematically the two-dimensional free surface problem of a jet of inviscid and incompressible fluid into a semi-infinite tube. The flow is considered to be irrotational.Where we take in the consideration the surface tension effect, the problem becomes very difficult because of the nonlinear condition on the free surface of the flow domain.This problem is also known as free boundary problems whose his mathematical formulation involves surfaces that have to be found as part of the solution. By using the integro-differential equation method, we solved numerically this problem for different values of the Weber number, and some typical profiles of the free surface of the jet are illustrated

Keywords: Integral equation ; Free-surface; Inviscid flow; Weber number.

1. THE INTRODUCTION

In this paper the problem of flow of a jet in a semi infinite tube is considered (See figure 1). The flow is steady irrotational, the flow is considered to be incompressible, inviscid and the effect of gravity is neglected, but we take in consideration the surface tension effect. The mathematical problem is defined by the number of Weber. When the effect of the surface tension is neglected, we can determine the exact solution by using the free streamline theory based on the conformal mapping theory[3]. In this case and when the effect of surface tension is considered, The problem becomes very difficult to solve analytically because of the nonlinear condition given by the Bernoulli equation on the free surface. which obliges us to use numerical techniques and methods that depend on conformal transformations to solve it. we use the integro-differential equation method and the Cauchy theorem. the main advantage of this method is to transform two-dimensional problems into unidimensional problems. To solve free surface problems, this method has been adopted by many previous authors ([1], [4], [5], [6], [7]). We were able to calculate the solution for different values of the Weber number and channel width. The results found confirm those found in [1].

2. MATHEMATICAL FORMULATION

The irrational flow along a semi-infinite rectangular channel is assumed. The fluid is inviscised and incompressible (see Figure 1)
The mathematical problem is to find the function ϕverified the following equation:

$$\Delta\phi = 0 \text{in the flow field,} \qquad (1)$$

Where ϕ is the velocity potential

$$\frac{\partial\phi}{\partial y} = 0 \text{in the walls } AB,\ CD \qquad (2)$$

$$\frac{\partial\phi}{\partial y} = 0 \text{in the wall } BC \qquad (3)$$

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 - \frac{T}{\rho}K = Cts \text{, on the free surface.} \qquad (4)$$

In this case ρ is the density, T is the surface tension, and K is the curvature of the free surface

$$\phi \longrightarrow Ux \qquad x \longrightarrow -\infty \qquad (5)$$

in which U the speed unit.

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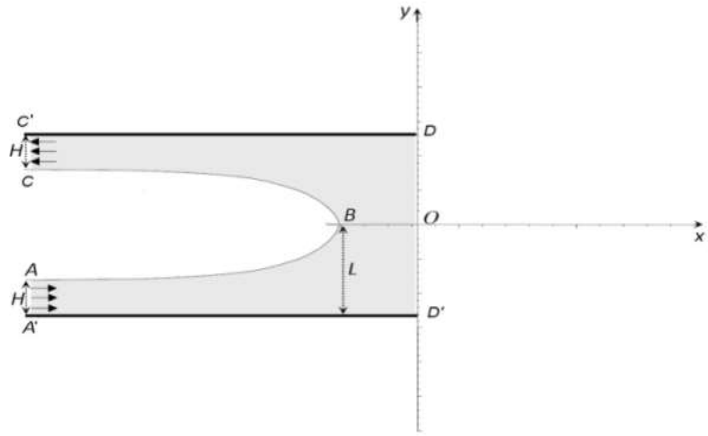

Figure 1: Sketch of the flow and the coordinates

In this way, the complex velocity W and the complex potential function f can be defined as:

$$f = \phi + i\psi,$$

$$W = \frac{df}{dz} = u - iv$$

Where u and v represent the horizontal and vertical components of the fluid velocity.
Without loss of generality, we choose $\psi = 0$ along the bottom $A'D'DC'$, then $\psi = 1$ on stream line ABC, and the configuration of the flow in the complex potential plane is sketched in Figure 2.

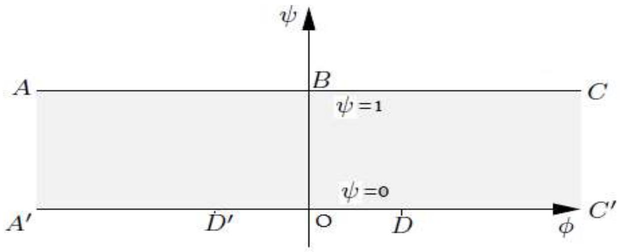

Figure 2: The complex potential f plane.

We are currently formulating the problem as an integral equation.
We define the function $\tau - i\theta$ as:

$$W = u - iv = exp(\tau - i\theta) \qquad (6)$$

Substitute (6) in (4), provides us with the final form of the Bernoulli equation that is necessary for numerical calculation

$$exp(2\tau) - 2We\ exp(\tau)\frac{\partial\theta}{\partial\phi} = 1 \qquad -\infty < \phi < +\infty \qquad (7)$$

Where $We = \frac{\rho UH}{T}$, is the Weber Number.
The kinematic boundary conditions on $A'D'$, $D'D$ and DC' can be expressed as:

$$u = 0\ on\ \psi = 0\ \ and - \infty < \phi < \phi_{D'} \qquad (8)$$

$$v = 0\ \ on\ \psi = 0\ and\ \ \phi_{D'} < \phi < \phi_D \qquad (9)$$

$$u = 0\ on\ \psi = 0\ \ and\ \ \phi_D < \phi < +\infty (10)$$

The function $\tau - i\theta$ is analytic in the strip $0 < \psi < 1$ and satisfy the conditions (7), (8), (9) and (10).

We map the flow domain onto the upper half of the ς -plane by the transformation

$$\varsigma = \alpha + i\beta = exp(-\pi f) \qquad (11)$$

The walls $A'D'$,$D'D$ and DC are mapped onto $-\infty < \alpha < 0$. The problem in the complex ς plane is illustrated in Fig.3.

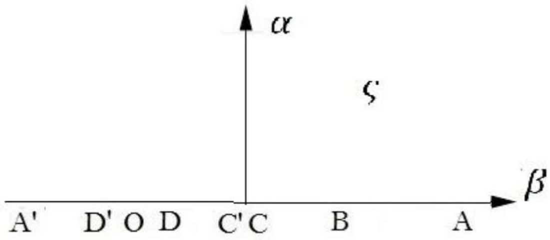

Figure 3: The complex ς-plane.

We introduce the curvilinear or contour integral of the$\tau - i\theta$function on a path closed by

$$\oint \frac{\tau(\varsigma) - i\theta(\varsigma)}{\varsigma - t} d\varsigma = 0 \qquad (12)$$

Where t is an image point of any point on the free surface $t \in ABC$. The path γ consists of a large semi-circular arc of radius R, centred at the origin, and the real axis with a circular indentation of radius ε about the point t See Figure 4.

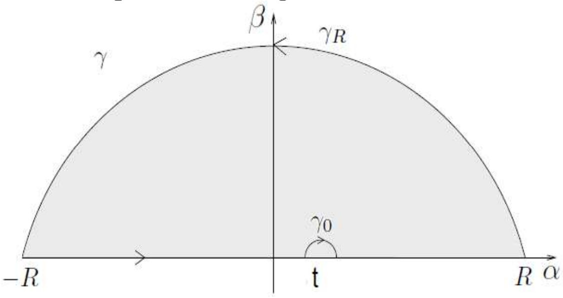

Figure 4: The complex ς-plane showing the contour.

When R tends to infinity, the contribution to the integral shape of the semicircle of radius R tends towards zero.
To the integrale in (12) is the principal value of Cauchy. Kinematic conditions (8), (9) , (10)and (11) imply

$$\theta(\alpha) = 0 \quad for \; -\infty < \alpha < \alpha_{D'} \qquad (14)$$

and

$$\theta(\alpha) = \frac{\pi}{2} \; for \alpha_{D'} < \alpha < \alpha_D \qquad (15)$$

and

$$\theta(\alpha) = \pi \qquad for \alpha_D < \alpha < 0 \qquad (16)$$

$$\tau'(\phi) = -\frac{1}{2}\left|\frac{e^{\pi\phi_D} - e^{\pi\phi_0}}{e^{\pi\phi_{D'}} - e^{\phi_0}}\right| - log\left|\frac{-e^{\pi\phi_0}}{e^{\pi\phi_D} - e^{\pi\phi_0}}\right| - \int_0^{+\infty} \frac{\theta'(\phi)e^{-\pi\phi}}{e^{\pi\phi} - e^{\pi\phi_0}} d\phi, \; -\infty < \phi < +\infty \qquad (18)$$

where $\tau'(\phi) = \tau(e^{-\pi\phi})$ and $\theta'(\phi) = \theta(e^{-\pi\phi})$

In (18) this integral equation is substituted to create an integro-differential equation which is then solved numerically.

3. NUMERICAL PROCEDURE

To solve the integro-differential non-linear equation obtained in the previous section. We use the numerical procedure and before that. The expression (18) is used to calculate τ along the free surface. It is necessary to have points, ϕ, along the free surface from which the values τ can be evaluated.
This is done by creating a discrete of the potential function-, on the free surface $0 < \phi < +\infty$
Let

$$\phi(I) = (I-1)\Delta, \quad I = 1, \dots\dots, N \qquad (18)$$

Where $\Delta > 0$

we assess the values $\tau^m(I)$of $\tau(\phi)$ at the midpoints

$$\phi^m(I) = \frac{\phi(I+1)+\phi(I)}{2}, \; I = 1, \dots\dots, N-1 \qquad (19)$$

by applying the trapezoidal rule, we obtain

$$\tau^m(I) = \frac{1}{2} log \left| \frac{e^{-\pi}-e^{\phi^m(I)}}{e^{\pi}-e^{\phi^m(I)}} \right| - log \left| \frac{e^{\phi^m(I)}}{e^{-\pi}+e^{\phi^m(I)}} \right| - \sum_1^N \frac{\theta(j)e^{\pi\phi(j)}\triangle w_i}{e^{-\pi\phi(j)}-e^{\phi^m(I)}}, I = 1, \dots., N-1 \qquad (20)$$

where $\theta(j) = \theta'(\phi(j))$ and w_j is the weighting function such that

$$\omega_j = \begin{cases} \frac{1}{2} & j = 1, N \\ 1 & otherwise \end{cases}$$

And

$$\frac{\partial\theta}{\partial\phi} = \frac{\theta_{I+1}-\theta_I}{\triangle} \qquad (21)$$

Substituting (20) into (7), for all the N midpoints, yields a system of N nonlinear algebraic equations for N unknowns $\theta(j), j = 1, \dots, N$. This system is also solved by Newton's method. The numerical calculations the previous, give a solution for the variables τ and θ. These variables are now used to obtain the equation of the free surface profile in the parametric form $x = x(\phi)$ and $y = y(\phi)$. Taking the real and imaginary parts of (6) we obtain

$$\frac{\partial x}{\partial\phi} = exp(-\tau)\, cos(\theta) \qquad (22)$$

And

$$\frac{\partial y}{\partial\phi} = exp(-\tau)\, sin(\theta) \qquad (23)$$

4. DISCUSSION OF RESULTS

Solution without surface tension effect

Numerical results are obtained when the Weber number tends towards infinity, i. e. when the surface tension effect tends towards zero, the system is reduced to :

$$exp(2\,\tau_I^m) = 1 \qquad I = 1, \dots\dots\dots, N \qquad (26)$$

We use the resolution method described above to resolve the system (26). We find that our results are identical to the results we have already found in the article [3]

Solution with tension effect

The same numerical procedure is used to solve the non-linear system (7) for different values of the Weber We number. The numerical calculation shows that there is a minimum value. We = 8 for which our numerical procedure converges

For $\mathrm{We} \geq 300$ all graphs describing the shape of the free surface are identical and coincide with the exact solution, so it can be said that surface tension after this value can be neglected. Figure 6 shows the different free surface profiles for $\mathrm{We} \geq 10$ and the few different values of H.

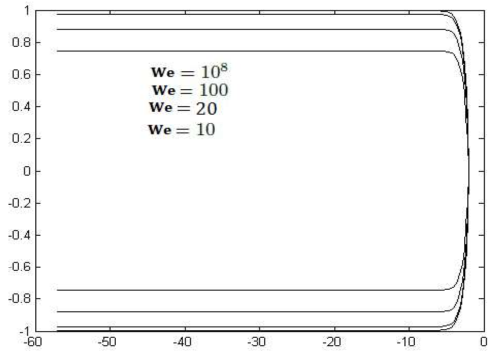

Figure 6: Free surface shapes for different Weber number values and different H

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STABILITY OF SOLUTIONS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATION BY USING LYAPUNOV FUNCTION

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ABSTRACT

In the present paper, several types of stability of the zero solution for a semilinear fractional-order system with exogenous input and Caputo fractional derivative have been studied using the Lyapunov function. In particular, conditional asymptotic stability and conditional Mittag-Leffler stability have been presented by introducing the Mittag-Leffler function of one and two parameters.

Keywords: Nonlinear fractional –order system; fractional calculus; conditional asymptotic stability; uniformly asymptotic stability; globally uniformly asymptotic stability.

1. INTRODUCTION

The fractional calculus generalizes the derivative and the integral of a function to the non-integer order. Several defnitions have been introduced by Grunwald-Letnikov, Caputo, Riemann-Liouville and others, in the next section we recall some of these definitions. For more details, interested authors advised to consult for example [11,19, 20].

In this work, we focused on the Mittag-Leffler function, one of the important special functions used in fractional calculus. Its importance is realized during the last one and a half decades due to its direct involvement in the problems of physics,biology, engineering and applied sciences. Mittag-Leffler function naturally occursas the solution of fractional-order differerential equations and fractional-order integral equations. Various properties of Mittag-Leffler functions are described in [5, 10, 15, 18]. Among the various results presented by various researchers, the important ones deal with Laplace transform and asymptotic expansions of these functions, which are directly applicable in the solution of differential equations and in the study of the behavior of the solution for small and large values of the argument.

Recently, fractional calculus was introduced to the stability analysis of nonlinear systems, see for example, [17] and many problems have been studied on this subject [7, 8, 13], where some basic results are obtained including stability theory. The question of stability is of main interest in physical and biological systems, such as the fractional Duffing oscillator [12], fractional predator-prey and rabies models [1]. Stability of nonlinear systems received increased attention due to its important role in areas of science and engineering. A large number of monograph and papers are devoted to the fractional nonlinear systems [3, 6, 14].

2. NOTES OF FRACTIONAL CALCULUS

Definition 2.1. ([19,20]). For a given interval $[a,b]$ in R, the Riemann-Liouville fractional integral of order $\alpha \succ 0$, of a function *u* in $L^1([a,b])$ is defined by:

$$I_a^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \quad t \in [a,b]. \ (1)$$

Definition 2.2. ([19]). For a given interval $[a,b]$ in R, the Caputo fractional derivative of order $\alpha \succ 0$, of a function u, is given by:

$${}_a^C D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau \ (2)$$

where $n-1 \prec \alpha \prec n.$

Definition 2.3. ([9,16]) The Mittag-Leffler function with two parameters is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, (3)$$

where $\alpha \succ 0,\ \beta \succ 0$ and $z \in C.$

3. NOTIONS PRELIMINARIES

Consider the following system of fractional differential equation with Caputo derivative

$${}_{t_0}^{C}D_t^{\alpha}x(t)=f(t,x),\quad t\geq t_0, (4)$$

where $0 \prec \alpha \prec 1$ and $f \in C(R_+ \times R^n, R^n).$

We will assume that for any initial data $(t_0,x_0)\in R_+\times R^n,$ the system (4) with the initial condition $x(t_0)=x_0$ has a solution $x(t;t_0,x_0)\in C^{\alpha}([t_0,+\infty),R^n).$ The purpose of the present paper is to study the stability of the system (4), for this fact let us suppose that in the rest of this paper that the origin $x=0$ is a point of equilibrium of the fractional-order system (4), that is $f(t,0)\equiv 0.$ Now, to get our results we need the following definitions:

Definition 3.1. The equilibrium point $x=0$ of the fractional-order system (4) is said to be

(a) Stable, if for every $\varepsilon \succ 0$ and $t_0 \in R_+$ there exists $\delta=\delta(\varepsilon,t_0)\succ 0$ such that for any $x_0 \in R^n$, the inequality $\|x_0\| \prec \delta$ implies $\|x(t,t_0,x_0)\| \prec \varepsilon,$ for $t \geq t_0.$

(b) Uniformly stable, if for every $\varepsilon \succ 0$ and $t_0 \in R_+$ there exists $\delta=\delta(\varepsilon)\succ 0$ such that for any $x_0 \in R^n$, and $\|x_0\| \prec \delta$ the inequality $\|x(t,t_0,x_0)\| \prec \varepsilon,$ holds for $t \geq t_0.$

(c) Uniformly attractive, if there exists $\beta \succ 0$ such that for every $\varepsilon \succ 0$ there exists $T=T(\varepsilon)\succ 0$ such that for any $t_0 \in R_+, x_0 \in R^n$ with $\|x_0\| \prec \beta$ the inequality $\|x(t,t_0,x_0)\| \prec \varepsilon,$ holds for $t \geq t_0+T.$

(d) Globally uniformly attractive if the definition (c) is verified for any $\beta \succ 0.$

(e) Uniformly asymptotically stable, if it is uniformly stable and uniformly attractive.

(f) Globally uniformly asymptotically stable, if it is uniformly stable and globally uniformly attractive.

Definition 3.2. We say that a continuous function $\phi : R_+ \to R_+$ is belongs to the class K if it is strictly increasing and $\phi(0)=0.$ If furthermore $\phi(t)\xrightarrow{t\to+\infty}+\infty,$ we say that φ belongs to the class K_∞. A continuous function $\psi : R_+ \to R_+$ is said to be class KL if $\psi(.,t)\in K.$

Definition 3.3. The nonlinear fractional-order system (5) is said to be conditional asymptotic stable, if for $\xi \succ 0$ such that for any input $\|\mu\| \leq \xi,$ there exist a class KL function ψ satisfying for each bounded initial condition $\|x(t_0)\|$ the solution $x(t)$ satisfies

$$\|x(t)\| \leq \psi(\|x(t_0)\|, t-t_0). (5)$$

Definition 3.4. The nonlinear fractional-order system (4) is said to be conditional Mittag-Leffler stable, if for $\xi \succ 0$ such that input $\|\mu\| \leq \xi,$ the solution $x(t)$ satisfies

$$\|x(t)\| \leq \left[k\|x(t_0)\|E_\alpha(\lambda(t-t_0)^\alpha)\right]^{\frac{1}{p}}, (6)$$

where k, p are two positive constants.

Lemme 3.1. Let us consider the following initial value problem for a nonhomogeneous fractional differential fractional equation with the Caputo fractional derivative of order $\alpha \in (0,1).$

$$ {}_{t_0}^{C}D_t^{\alpha} y(t) - \lambda y(t) = g(t), \quad t \ge t_0, \qquad (7) $$
$$ y(t_0) = y_0 $$

Problem (7) was studied by Podlubny in [19] and its solution is given by:

$$ y(t) = y_0 E_{\alpha}(\lambda(t-t_0)^{\alpha}) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) g(s) ds. \quad (8) $$

4. STABILITY RESULTS

Theorem 4.1. Assume that there exist a function $V \in C(R_+ \times R^n, R_+)$ which has Caputo fractional derivative of order α for all $t \ge t_0$, such that $V(t,0) \equiv 0$ and a class K functions α_1, α_2 satisfying

$$ \alpha_1(\|x\|) \le V(t, x(t)) \le \alpha_2(\|x\|), \quad \forall t \ge t_0, \forall x \in R^n, \quad (9) $$

$$ {}_{t_0}^{C}D_t^{\alpha} V(t, x(t)) \le -(k - \alpha_1(\|\mu\|))\alpha_2(\|x\|), \quad \alpha_1(\|\mu\|) \prec k. \qquad (10) $$

If the input $\|\mu\| \le \xi$ is satisfied, then $x = 0$ is uniformly asymptotically stable. In addition, if α_1, α_2 are two class K_{∞} functions, then $x = 0$ is globally uniformly asymptotically stable.

Proof of Theorem4.1. First, we show that $x = 0$ is uniformly stable. The condition (10) implies that there exist a nonnegative function $h(t)$ satisying

$$ {}_{t_0}^{C}D_t^{\alpha} V(t, x(t)) \le -h(t), \quad \forall t \ge t_0. \quad (11) $$

From (11), it follows that:

$$ V(t, x(t)) \le V(t_0, x_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} h(s) ds, $$
$$ \le V(t_0, x(t_0)), \quad (12) $$

then, the condition (9) and inequality (12) leads to:

$$ \alpha_1(\|x(t)\|) \le V(t_0, x_0). \quad (13) $$

Now, for any $\varepsilon \succ 0$, we can find $\delta = \delta(\varepsilon)$ such that $\alpha_2(\delta) \prec \alpha_2(\varepsilon)$. Let $x_0 \in R^n$ such that $\|x_0\| \prec \delta$. By using (10) and (13), we obtain that: $\alpha_1(\|x(t)\|) \le \alpha_2(\|x_0\|) \prec \alpha_2(\delta) \prec \alpha_1(\varepsilon)$.

Since $\alpha_1 \in K$, then we have: $\|x(t)\| \prec \varepsilon, \quad \forall t \ge t_0$. Therefore, $x = 0$ is uniformly stable.

Now, we show that $x = 0$ is uniformly attractive. Let r be a positive number such that $\alpha_2(\|x_0\|) \prec r$. From the assumption $\|\mu\| \le \xi$ and the conditions (9) and (10) it follows that:

$$ {}_{t_0}^{C}D_t^{\alpha} V(t, x(t)) \le -cV(t, x(t)), \quad (14) $$

The inequality (14) implies that, there exist a nonnegative function $g(t)$ satisfying:

$$ {}_{t_0}^{C}D_t^{\alpha} V(t, x(t)) \le -cV(t, x(t)) - g(t). \quad (15) $$

Then, we have

$$ V(t, x) = V(t_0, x_0) E_{\alpha}(-c(t-t_0)^{\alpha}), \quad \forall t \ge t_0. \quad (16) $$

A combination of (9) and (16) gives:

$$ \alpha_1(\|x(t)\|) \le \alpha_2(\|x_0\|) E_{\alpha}(-c(t-t_0)^{\alpha}), \quad (17) $$

that is to say:

$$ \alpha_1(\|x(t)\|) \le r E_{\alpha}(-c(t-t_0)^{\alpha}). \quad (18) $$

From (18), it follows that :

$$ \|x(t)\| \le \alpha_1^{-1}(r E_{\alpha}(-c(t-t_0)^{\alpha})). \quad (19) $$

Since $\lim_{\lambda \to +\infty} E_{\alpha}(-c\lambda^{\alpha}) = 0$, then

$\lim\limits_{\lambda \to +\infty} \alpha_1^{-1}(rE_\alpha(-c\lambda^\alpha)) = 0, (20)$

(because $\alpha_1^{-1}(0) = 0$). Hence, we have for all $\varepsilon \succ 0$, there exist $T = T(\varepsilon) \succ 0$ such that :

$\alpha_1^{-1}(rE_\alpha(-c(t-t_0)^\alpha)) \prec \varepsilon, \quad \forall t - t_0 \geq T,$

which means that:

$E_\alpha(-c(t-t_0)^\alpha) \prec \dfrac{\alpha_1(\varepsilon)}{r}, \quad \forall t - t_0 \geq T. (21)$

Thus, from (19) and (21), it follows that: $\|x(t)\| \prec \varepsilon, \quad \forall t \geq t_0 + T.$

The last inequatlity shows that $x = 0$ is uniformly attractive. Therefore, $x = 0$ is uniformly asymptotically stable. Now, suppose that $\alpha_1, \alpha_2 \in K_\infty$. In view of (17), it follows that:

$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x_0\|)E_\alpha(-c(t-t_0)^\alpha)), \forall t \geq t_{0.}.(22)$

Let $\forall \varepsilon \succ 0$ and $\xi \succ 0$ such that $\|x_0\| \prec \xi.$ From (22), it follows that:

$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\xi)E_\alpha(-c(t-t_0)^\alpha)), \forall t \geq t_{0.}.(23)$

Then, by using (20), we find that there exist $T = T(\varepsilon, \xi)$ such that:

$E_\alpha(-c(t-t_0)^\alpha)) \prec \dfrac{\alpha_1(\varepsilon)}{\alpha_2(\xi)}, \forall t - t_0 \geq T, (24)$

hence, from (23) and (24) we obtain: $\|x(t)\| \prec \varepsilon, \quad \forall t \geq t_0 + T,$

this inequality means that $x = 0$ is globally uniformly attractive. Therefore, $x = 0$ is globally uniformly asymptotically stable.□

Theorem 4.2. Assume that there exist a function $V \in C(R_+ \times R^n, R_+)$ which hasCaputo fractional derivative of order α for all $t \geq t_{0.}$ and a class K_∞ functions α_1, α_2 satisfying

$\alpha_1(\|x\|) \leq V(t, x(t)) \leq \alpha_2(\|x\|), \quad \forall t \geq t_0, \forall x \in R^n, (25)$

${}_{t_0}^{C}D_t^\alpha V(t, x(t)) \leq -(k - \alpha_1(\|\mu\|))V(t, x(t), \quad \alpha_1(\|\mu\|) \prec k. (26)$

then $x = 0$ is conditional asymptotically stable.

Proof of Theorem4.2. In view of the condition (26) and the assumption $\|\mu\| \leq \xi$, we find that:

${}_{t_0}^{C}D_t^\alpha V(t, x(t)) \leq -(k - \alpha_1(\xi))V(t, x(t)),$ then there exist a nonnegative continuous function $h(t)$ such that ${}_{t_0}^{C}D_t^\alpha V(t, x(t)) = -cV(t, x(t)) - h(t).$

From Lemma 3.1, it follows that for $t \geq t_0$:

$V(t, x) \leq V(t_0, x_0)E_\alpha(-c(t-t_0)^\alpha) (27)$

Therefore, the inequality (27) and the condition (25) leads to:

$\alpha_1(\|x(t)\|) \leq \alpha_2(\|x_0\|)E_\alpha(-c(t-t_0)^\alpha),$

this means that:

$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x_0\|)E_\alpha(-c(t-t_0)^\alpha)). (28)$

Then (28) gives: $\|x(t)\| \leq \psi(\|x_0\|, t - t_0)$. Thus, $x = 0$ is conditional asymptotically stable. □

Theorem 4.3. Assume that there exist a function $V \in C(R_+ \times R^n, R_+)$ which has Caputo fractional derivative of order α for all $t \geq t_0$ and a class KL function α_1 satisfying

$c_1\|x\|^p \leq V(t, x(t)) \leq c_2\|x\|^p, \quad \forall t \geq t_0, \forall x \in R^n, (29)$

${}_{t_0}^{C}D_t^\alpha V(t, x(t)) \leq -(k - \alpha_1(\|\mu\|))\|x\|^p, \quad \alpha_1(\|\mu\|) \prec k. (30)$

where c_1, c_2, p and k are positive constants. Then $x=0$ is conditional Mittag-Leffler stable.

Proof of Theorem4.3. By the conditions (29),(30) and the assumption $\|\mu\| \le \xi$, we find that:

$${}_{t_0}^{C}D_t^{\alpha}V(t,x(t)) \le -\frac{k-\alpha_1(\xi)}{c_2}V(t,x(t)). \qquad (31)$$

Inequality (31) means that there exist a nonnegative function $h(t)$ such that:

${}_{t_0}^{C}D_t^{\alpha}V(t,x(t)) \le -cV(t,x(t)) - h(t),(32)$ By using Lemma 3.1, it follows that for $t \ge t_0$: $\|x(t)\| \le \left[M\|x(t_0)\|E_\alpha(-c(t-t_0)^\alpha)\right]^{\frac{1}{p}}$, where M is a positive constant. Then point $x=0$ is conditional Mittag-Leffler stable. □

5. ILLUSTRATIVE EXAMPLE

Before giving some illustrative examples, we need the following auxiliary lemma:

Lemme 5.1. ([2]). For any differentiable vector $x(t) \in R^n$ and any time instant $t \ge t_0$, we have:

$$\frac{1}{2}{}_{t_0}^{C}D_t^{\alpha}\left[x^T(t)x(t)\right] \le x^T(t){}_{t_0}^{C}D_t^{\alpha}x(t), \quad \forall \alpha \in (0,1).$$

Now, In all that follows, we consider $x(t) = (x_1(t), x_2(t), x_3(t)) \in R^3$, and $\|x(t)\|$ stands for its Euclidean norm: $\|x(t)\| = (\sum_{i=1}^{3} x_i^{\,2})^{\frac{1}{2}}$ and $0 \prec \alpha \prec 1$.

Example 5.1. Consider the following fractional-order system:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x_1 = -4x_1 + e^{-t}\cos(x_2)x_1 \\ {}_{t_0}^{C}D_t^{\alpha}x_2 = -4x_2 + \dfrac{\sin(x_3)}{1+t^2}x_2 \\ {}_{t_0}^{C}D_t^{\alpha}x_3 = -4x_3 + \sin(x_2)x_3 \end{cases} \qquad (33)$$

: $V(t,x) = \dfrac{x_1^{\,2} + x_2^{\,2} + x_3^{\,2}}{4}$ with the input $\|\mu\| \le \xi$. By using Lemma 5.1, we have:

$${}_{t_0}^{C}D_t^{\alpha}V(t,x(t;t_0,x_0)) \le \frac{1}{2}\left[x_1(t;t_0,x_0){}_{t_0}^{C}D_t^{\alpha}x_1(t;t_0,x_0) + x_2(t;t_0,x_0){}_{t_0}^{C}D_t^{C}x_2(t;t_0,x_0)\right.$$

$$\left.+ x_3(t;t_0,x_0){}_{t_0}^{C}D_t^{\alpha}x_3(t;t_0,x_0)\right] = -6V(t,x(t;t_0,x_0)).$$

Then, it is enough to choose $\alpha_1(\|\mu\|) \prec k \le \alpha_1(\|\mu\|) + 6$. Now, all assumptions of the Theorem 4.2 are satisfied, therefore $x=0$ is conditional asymptotically stable.

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NUMERICAL STUDY OF STAGNATION POINT FLOW OVER A SPHERE WITH GO/ WATER AND KEROSENE OIL BASED MICROPOLAR NANOFLUID

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ABSTRACT

In this article, the mixed convection boundary layer flow in micropolar nanofluids at the lower stagnation point of a solid sphere in a stream flowing vertically upwards has been studied numerically for both issues of a heated and cooled solid sphere with a constant surface heat flux. Grephene oxide nanoparticle suspended in two different types of fluids namely water and kerosene oil. The governing partial differential equations including continuity, momentum and energy have been reduced to ordinary differential equations ones and solved via an implicit finite-difference scheme known as the Keller-box method. Numerical solutions are taken out for temperature profiles, velocity profiles, angular velocity profiles, with different values of the parameters, namely, the nanoparticle volume fraction χ and the mixed convection parameter λ . itis found that GOwater has higher in temperature compared with GOkerosene oil

*Keywords:*Mixed Convection, stagnation point,MicropolarNanofluid, Solid Sphere.

1. INTRODUCTION:

A nanofluid is a heat-transfer fluid[1] containing nanoparticles with a size smaller than 100 nm such as oxides, metals and carbides [2]. Common base fluids comprise water oil and ethylene glycol[3], The nanoparticles have a unique chemical and physical properties, while compared only to base fluid, will increase the efficiency of the thermal conductivity and the convective heat-transfer coefficient [4]. Nanofluids have many properties that make them potentially useful in several applications in heat transfer, such as microelectronics, fuel cells, pharmaceutical processes, and hybrid-powered engines. Buongiorno, [5] published an article on the convective transport in nanofluids. The nanofluid flow inside a two-sided lid-driven differentially heated square cavity is studied numerically by Tiwariet al, [6]. The nanofluids used to acquire optimum thermal properties at the lowest volume fraction of nanoparticles in the base fluid by Godson et al, [7]. Kandelousi, [8] also considered the nanofluid flow and heat transfer through a permeable channel. Haqet al, [9] studied the slip effect on heat transfer nanofluid flow past a stretching surface.Several references have on nanofluid as in the universal book by Das et al, [1], and many studies that have been conducted to boost the heat-transfer characteristics technique by nanofluids, including those by [10-16].

The classical Navier-Stokes theory described the flow properties of non-Newtonian materials, but this theory was not suitable to describe microrotations, certain microscopic effects growing from the local structure of fluid elements, and some naturally arising fluids, which are known as micropolar or thermomicropolar fluids. Micropolar fluid theory and its dilation to thermomicropolar fluids were initially introduced by Eringen, [17]. Further, many physicists, engineers and mathematicians have been studied on the micropolar fluid to conclude the different results related to flow problems. Hassanienet al[18]presented the boundary layer flow and heat transfer from a stretching sheet to a micropolar fluid. Papautskyet al, [19] investigated the laminar fluid behaviour in microchannels using micropolar fluid theory. Nazaret al, [20]consideredstagnation point flow of a micropolar fluid towards a stretching sheet.Exact solutions are obtained by the Laplace transform technique for the unsteady flow of a micropolar fluid by Sheriefet al[21]. Hussananet al[22]described the microrotation, temperature, velocity and concentration are considered. Hussananet al[23]explained the unsteady natural convection flow of a micropolar fluid on a vertical plate oscillating in its plane with Newtonian heating condition. Free convection boundary layer flow of micropolar fluid on a solid sphere with convective boundary conditions was considered by Alkasasbehet al,[24]. Alkasasbeh,[25] explores the heat transfer magnetohydrodynamic flow of micropolar Casson fluid on a horizontal circular cylinder with thermal radiation. Natural convection on boundary layer flow of Cu-water and Al_2O_3-water micropolar nanofluid about a solid sphere investigated by Swalmehet al, [26] and micropolar forced convection flow over moving surface under magnetic field was inspected byWaqaset al,[27].

The aim of this paper is to study the mixed convection boundary layer flow over a solid sphere in a micropolarnanofluid with constant surface heat flux. graphene oxide (GO) in two based micropolarnanofluids (water and kerosene oil) havebeen considered in the present investigation. The boundary-layer equations are solved numerically via efficient implicit finite-difference scheme known as the Keller-box method, as displayed by [28]. The effect of the nanoparticle volume fraction parameter, the mixed convection parameter and micro-rotation parameter on temperature, velocity and angular velocity at the lower stagnation point of the sphere are discussed and explained in the tables and figures.

2. BASIC EQUATIONS

Consider the impermeable solid sphere of radius a, which is placed in an incoming stream of micropolarnanofluid with an undisturbed free-stream velocity U_∞ and constant temperature T_∞, with steady mixed convection boundary-layer flowIt is also supposed that the surface of the sphere is maintained at a constant temperature, T_w with $T_w > T_\infty$ for a heated sphere (assisting flow) and $T_w < T_\infty$ for a cooled sphere(opposing flow).

The basic steady dimensionalmomentum and energy equations for micropolarnanofluid,which are defined byTiwari and Das [6].and Swalmeh et al.[26]

$$\frac{\rho_f}{\rho_{nf}}\left(D(\chi)+K\right)f''' + 2ff'' - (f')^2 + \frac{1}{\rho_{nf}}\left(\chi\rho_s\left(\frac{\beta_s}{\beta_f}\right)+(1-\chi)\rho_f\right)\lambda\theta + \frac{\rho_f}{\rho_{nf}}K\frac{\partial h}{\partial y} + \frac{9}{4} = 0, \tag{1}$$

$$\frac{1}{\Pr}\left[\frac{k_{nf}/k_f}{(1-\chi)+\chi\left(\rho c_p\right)_s/\left(\rho c_p\right)_f}\right]\theta'' + 2f\theta' = 0, \tag{2}$$

$$\frac{\rho_f}{\rho_{nf}}\left(D(\chi)+\frac{K}{2}\right)h'' + 2f\,h' - f'h - \frac{\rho_f}{\rho_{nf}}K\left(2h+f''\right) = 0. \tag{3}$$

alongwiththe boundary conditions

$$f(0) = f'(0) = 0,\ \theta'(0) = -1,\ h(0) = -\frac{1}{2}f''(0) \text{ as } y = 0,$$

$$f' \to \frac{3}{2},\ \theta \to 0,\ h \to 0 \text{ as } y \to \infty, \tag{4}$$

where the primes denote differentiation with respect to y.[28].

3. RESULTS AND DISCUSSIONS

Equations (1)–(3) subject to the boundary conditions (4) have been solved numericallyby using an efficient implicit finite-difference scheme known as the Keller-box method, along with Newton's linearization technique as described by [28]for verious values of parameters: mixed convection parameter λ, the micro-rotation parameter$K$, and the nanoparticle volume fraction χ on temperature, velocity and angular velocity fields, at the lower stagnation point of a solid sphere, $x \approx 0$, for both the assisting $(\lambda > 0)$ and opposing ($\lambda < 0$) flow cases

Figures 1to6display the characteristics of the nanoparticle volume fraction χ and the micro-rotation parameter K on the temperature profiles, the velocity profiles, and the angular velocity respectively, of GO in water and kerosene oil at the lower stagnation point of the sphere, $x \approx 0$. It can be seen that when the nanoparticle volume fraction χ and the micro-rotation parameter K increase, the velocity profiles and the angular velocity profiles decrease, but the temperature profilesincrease. Besides that, it is also noticed that GO water has a higher temperature, velocity and angular velocity compared with GO kerosene oil for every value ofthe nanoparticle volume fraction χ and the micro-rotation parameter K .

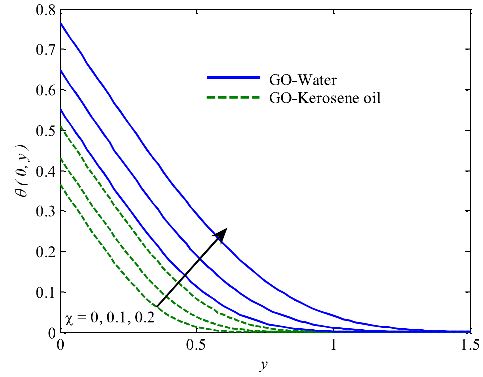

Fig.1. Temperature profiles at $x \approx 0$ using GO in water and kerosene oil-basednanofluids, for various values of χ, when $\lambda = 3$ and $K = 0.3$.

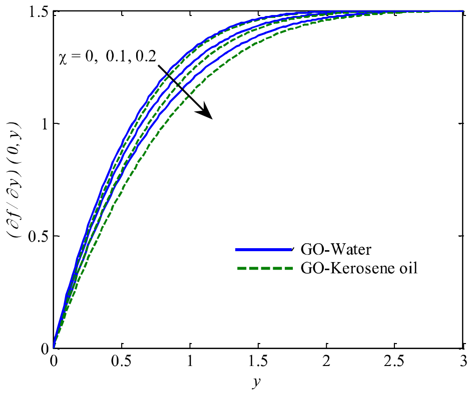

Fig.2. Velocity profiles at $x \approx 0$ using GO in water and kerosene oil-basednanofluids, for various values of χ, when $\lambda = 3$ and $K = 0.3$

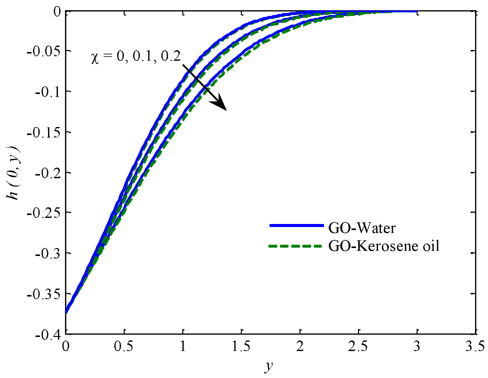

Fig.3. Angular velocity profiles at $x \approx 0$ using GO in water and kerosene oil-basednanofluids, for various values of χ, when $\lambda = 3$ and $K = 0.3$

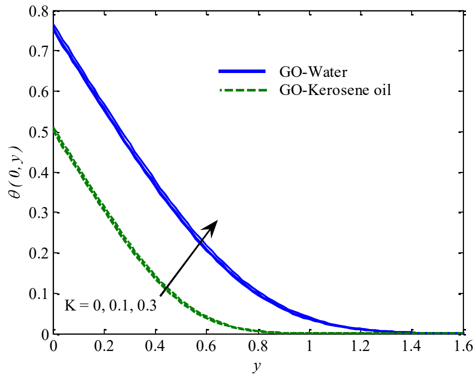

Fig.4. Temperature profiles at $x \approx 0$ using GO in water and kerosene oil-basednanofluids, for various values of K, when $\lambda = 3$ and $\chi = 0.2$

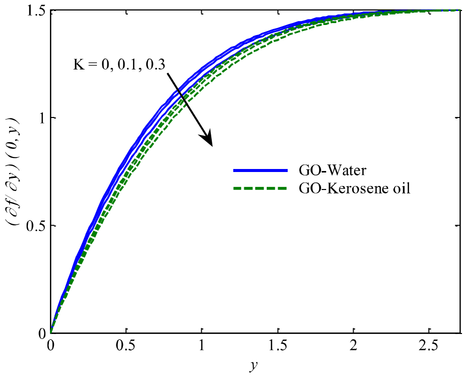

Fig.5. velocity profiles at $x \approx 0$ using GO in water and kerosene oil-basednanofluids, for various values of K, when $\lambda = 3$ and $\chi = 0.2$.

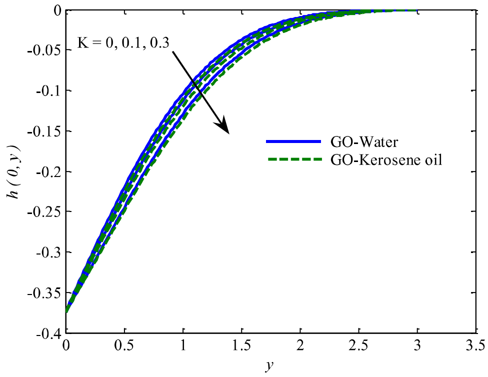

Fig.6 Angular velocity profiles at $x \approx 0$ using GO in water and kerosene oil-basednanofluids, for various values of K, when $\lambda = 3$ and $\chi = 0.2$.

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4. CONCLUSIONS

In this paper, we have numerically studied the mixed convection boundary-layer flow about solid sphere in amicropolarnanofluid with constant surface heat flux. We discussed into the effects of the mixed convectionparameter λ, the nanoparticle volume fraction χ, the micro-rotation parameter K, and nanoparticles GO suspended in two based fluids, such as water and kerosene oil. The problem is modelled and then solved via Keller box method. From this study, we could conclude the following conclusions:

i. The GO water has a higher temperature, velocity and angular velocity compared with GO kerosene oil for every value of parameters χ and K.

ii. The GO kerosene oil has a lower temperature compared with GO water for every value of. λ.

iii. The GO water has a higher velocity and angular velocity compared with GO kerosene oil for every value of parameter λ, but the opposite happens when the case of the cooled sphere ($\lambda < 0$).

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DYNAMICAL PROPERTIES OF SOLUTIONS IN A 3-D LOZI MAP

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ABSTRACT

In this letter, the existence of some properties of solutions in 3-D Lozi map is presented, that the results have been confirmed by simple rigorous mathematical analysis methods.

Keywords: 3-D Lozi map, Unbounded orbits, Global attractors, solutions of the 3-D Lozi map.

1. INTRODUCTION

In literature [2], the three-dimensional Hénon map is quadratic map with constant Jacobian matrix determinant, and its inverse map is quadratic, and the coordinates are not decoupled by the action of the map. Several researchers have defined and studied quadratic 3-D chaotic maps such as with quadratic inverse and constant Jacobi [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] such as the simplest 3-D quadratic map studied in [1] and given by

$$H(x,y,z) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 + bz + ax^2 \\ x \\ y \end{pmatrix} \quad (1)$$

Where $(x,y,z) \in \mathbb{R}^3$ and $(a,b) \in \mathbb{R}^2$ are map parameter, $a \neq 0$ and $b \neq 0$. The chaotic attractor in Fig.1 exhibited by the 3-D Hénon map (1) is very similar to the attractor of the famous 2-D Hénon map [14, 15] and are obtained from a period-doubling bifurcation route to chaos.

The 3-D Lozi map (2) is a simplification form of the 3-D Hénon map (1), obtained from a simple modification the quadratic nonlinear term x^2 is replaced by the piecewise term $|x|$. Then the form of the 3-D Lozi map (2) is given by

$$h(x,y,z) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 + bz + a|x| \\ x \\ y \end{pmatrix} \quad (2)$$

Furthermore, recent publication [13] show that while varying the parameter a or b the attractor of the 2-D Lozi map [16] and the attractor of the 3-D Lozi map (2) in Fig.4 are very similar and are obtained from a border-collision bifurcation route to chaos.
On the other hand we can transform the 3-D Lozi map (2) into a third order difference equation: Let (x_t, y_t, z_t), $i = 1,2,\ldots$ be a trajectory of the map (2) and we suppose $x = x_t$, $y = x_{t-1}$ and $z = x_{t-2}$ then the map (2) can be written as

$$x_{t+1} = 1 + bx_{t-2} + a|x_t| \quad (3)$$

we remark that the space can be separated into two linear areas are defined by

$$\begin{cases} \Sigma_1 = (x,y,z) \in \mathbb{R}^3 : x > 0 \\ \Sigma_2 = (x,y,z) \in \mathbb{R}^3 : x < 0 \end{cases}$$

In the two areas $\Sigma_1$and Σ_2 , map (3) can be rewritten as follows

$$x_{t+1} = \begin{cases} 1 + bx_{t-2} + ax_t \text{if } x \in \Sigma_1 \\ 1 + bx_{t-2} - ax_t \text{ if } x \in \Sigma_2 \end{cases}$$

This paper studies the existence of some properties of solutions of the 3-D Lozi map (3) such as, stability, attractivity, unboundedess and exact formula of solutions.

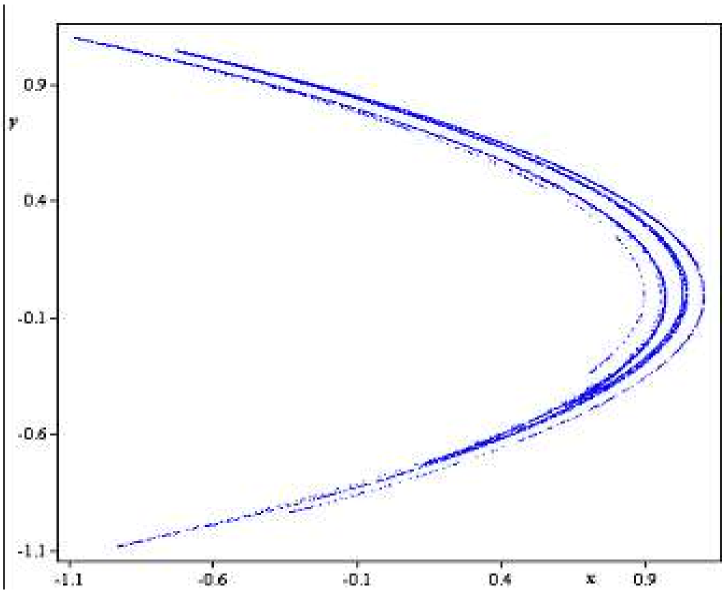

Figure 1: Chaotic attractor obtained in *xy*-plan from the 3-D Hénon map (1) for $a = -1.65$ and $b = 0.1$.

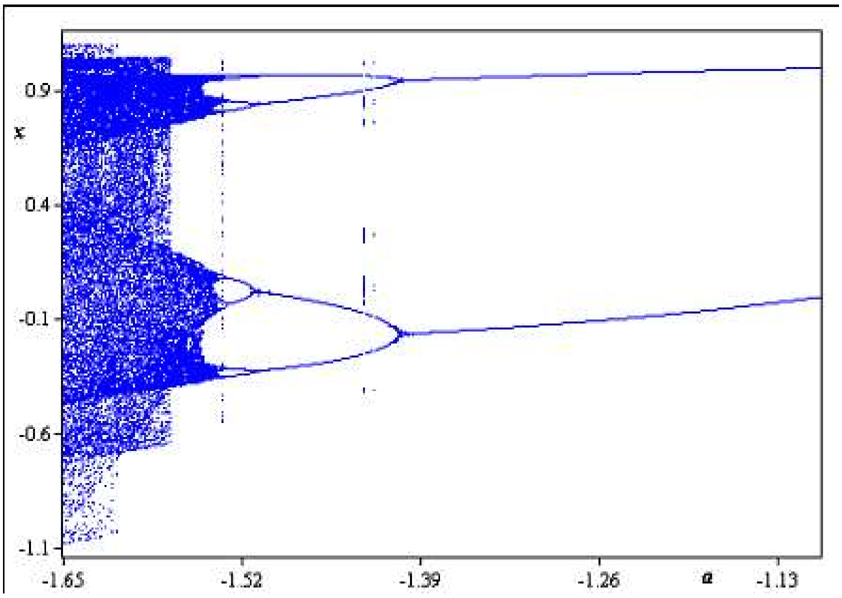

Figure 2: Bifurcation diagram of the 3-D Hénon map (1) obtained for $b = 0.1$ and $-1.65 \le a \le 1.1$.

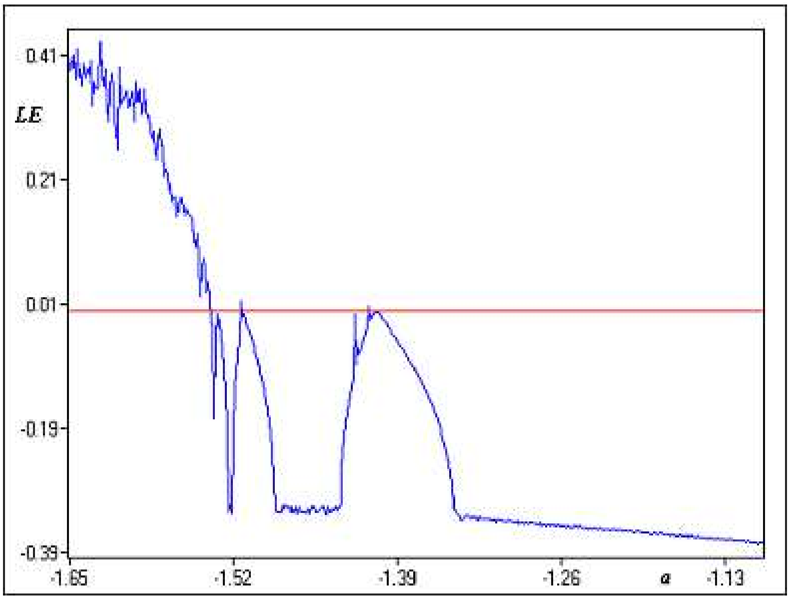

Figure 3: Variation of the largest Lyapunov exponent of the 3-D Hénon map (1) for $b = 0.1$ and $-1.65 \leq a \leq 1.1$.

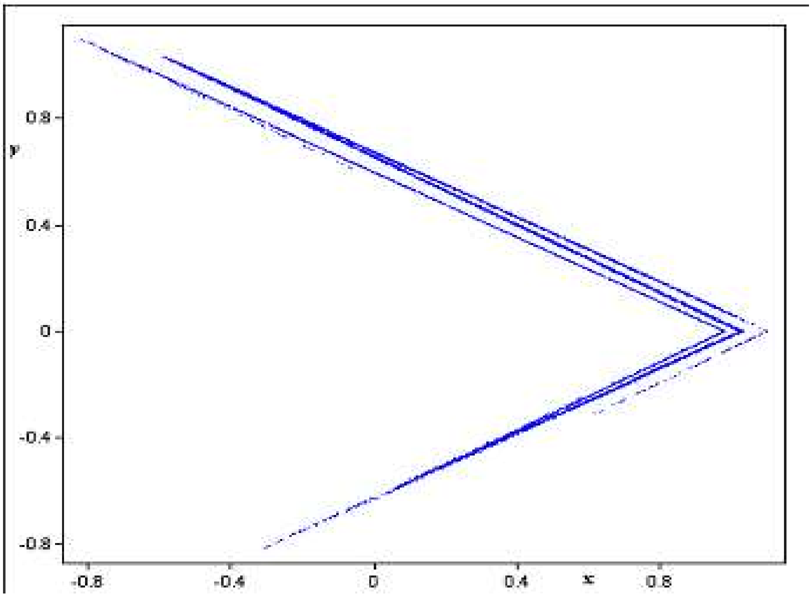

Figure 4: Chaotic attractor obtained in *xy*-plan from the 3-D Lozi map (2) for $a = -1.65$ and $b = 0.1$.

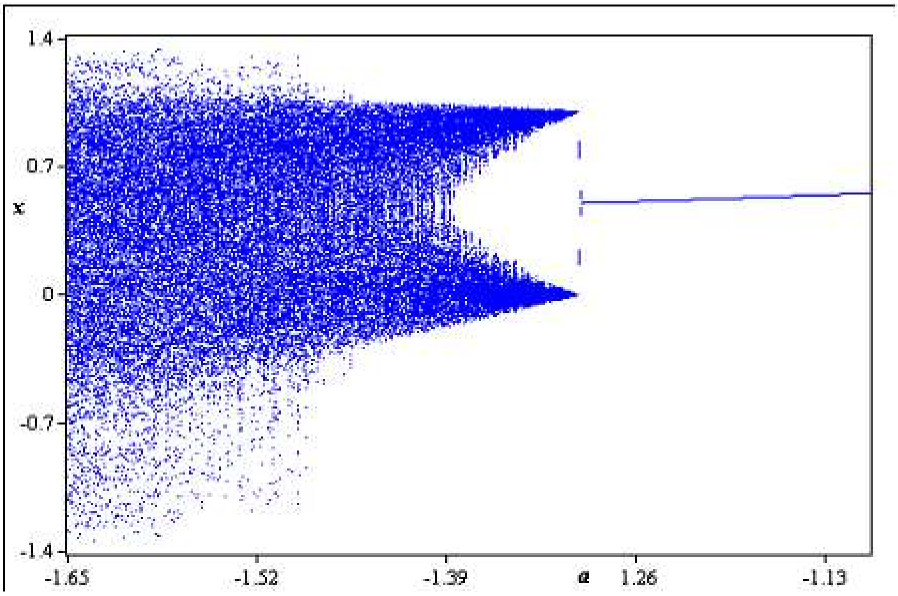

Figure 5: Bifurcation diagram of the 3-D Lozi map (2) obtained for $b = 0.1$ and $-1.65 \leq a \leq 1.1$.

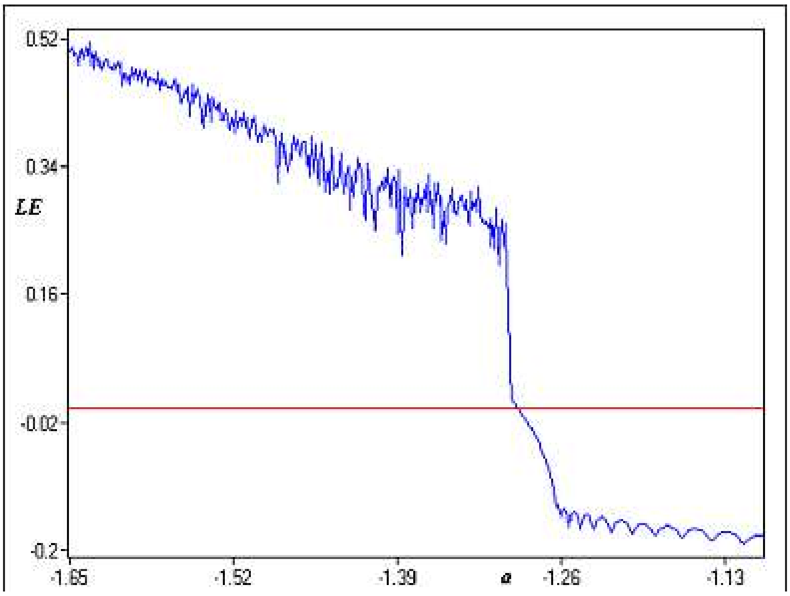

Figure 6: Variation of the largest Lyapunov exponent of the 3-D Lozi map (2) for $b = 0.1$ and $-1.65 \leq a \leq 1.1$.

2. STABILITY CONDITIONS OF SOLUTIONS OF THE 3-D LOZI MAP

In this section we investigate the local stability of solutions of the 3-D Lozi map (3).

Theorem 2.1. For all values of the map parameters $(a, b) \in \mathbb{R}^2 : b < 1 - a$ and $b > 1 + a$, the 3-D Lozi map (3) has two fixed points, and they are given by

$$S_1 = \frac{-1}{b+a-1}(1,1,1) \text{ and } S_2 = \frac{-1}{b-a-1}(1,1,1).$$

Proof. The fixed point of the 3-D Lozi map (3) is the real solutions of

$$1 + bx + a|x| = x$$

If $x \in \Sigma_1$ we have $(b + a - 1)x = -1$ then one has, $x = \frac{-1}{b+a-1}$ with $b < 1 - a$ then we have the fixed point S_1. If $x \in \Sigma_2$ we have $(b - a - 1)x = -1$ then one has, $x = \frac{-1}{b-a-1}$ with $b > 1 + a$. then we have the fixed point S_2.

Theorem 2.2. If $a > 0$ and $b > 0$, then the fixed point S_1 of the 3-D Lozi map (3) is locally asymptotically stable if $a + b < 1$.

Proof. Let $f : \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x, y, z) = 1 + ax + bz$, we have $f_x(x, y, z) = a$, $f_y(x, y, z) = 0$ and $f_z(x, y, z) = b$. If $x \in \Sigma_1$ and $a > 0$, $b > 0$ the linearized equation of the 3-D Lozi map (3) associated with this fixed point S_1 is, $y_{t+1} = f_x(x, y, z)y_t + f_y(x, y, z)y_{t-1} + f_z(x, y, z)y_{t-2}$.
or

$$y_{t+1} - ay_t - by_{t-2} = 0 (4)$$

according to the Theorem available in [15] the 3-D Lozi map (3) is asymptotically stable if

$$|a| + |b| < 1 (5)$$

For $a, b > 0$ and from (5) we obtain $a + b < 1$.

Theorem 2.3. If $a < 0$ and $b < 0$, then the fixed point S_2 of the 3-D Lozi map (3) is not locally asymptotically stable if $b - a > 1$.

Proof. Let $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ be a function defined by $g(x, y, z) = 1 - ax + bz$, we have $g_x(x, y, z) = -a$, $g_y(x, y, z) = 0$ and $g_z(x, z) = b$. If $x \in \Sigma_2$ and $a < 0$, $b < 0$ the linearized equation of the 3-D Lozi map (3) associated with this fixed point S_2 is, $y_{t+1} = g_x(x, y, z)y_t + g_y(x, y, z)y_{t-1} + g_z(x, y, z)y_{t-2}$.

or

$$y_{t+1} + ay_t - by_{t-2} = 0 (6)$$

according to the Theorem available in [15] the 3-D Lozi map (3) is not asymptotically stable if

$$|a| - |b| > 1 (7)$$

For $a < 0$, $b < 0$ and from (7) we obtain $b - a > 1$.

3. ATTRACTIVITY OF SOLUTIONS OF THE 3-D LOZI MAP

In this section, we aim to examine the global attractivity of solutions of the 3-D Lozi map (3):

Theorem 3.1. If $x \in \Sigma_1$ and $a > 0$, $b < 0$ then the fixed point S_1 of the 3-D Lozi map (3) is global attractor.

Proof. Let α, β $(\alpha < \beta)$ are a real numbers and consider that V: $[\alpha,\beta]^2 \rightarrow [\alpha,\beta]$ be a function defined by $V(x, z) = 1 + bz + ax$ then it is easy to see that the function $V(x, z)$ is increasing in x if $a > 0$ and decreasing in z if $b < 0$. Suppose that (m, M) is a solution of the system $M = V(M, m)$ and $m = V(m, M)$ then we have that $M = 1 + bm + aM$ and $m = 1 + bM + am$ therefore $(1 - a)M = 1 + bm$ and $(1 - a)m = 1 + bM$,subtracting we have that $(1 - a)(M - m) = b(m - M)$ since $b < 1 - a$ we obtain $m = M$. According to the result available in [18] that the S_1 is a global attractor of the 3-D Lozi map (3).

Theorem 3.2. If $x \in \Sigma_2$and $a, b < 0$ then the fixed point S_2 of the 3-D Lozi map (3) is global attractor.

Proof. Let α, β ($\alpha < \beta$) are a real numbers and consider that W: $[\alpha,\beta]^2 \rightarrow [\alpha,\beta]$ be a function defined by $W(x, z) = 1 + bz + ax$ then it is easy to see that the function $W(x, z)$ is increasing in x if $a < 0$ and decreasing in z if $b < 0$. Suppose that (m, M) is a solution of the system $M = W(M, m)$ and $m = W(m, M)$ then we have that $M = 1 + bm - aM$ and $m = 1 - bM + am$ therefore $(1 + a)M = 1 + bm$ and $(1 + a)m = 1 + bM$,subtracting we have that $(1 + a)(M - m) = b(m - M)$ since $b > 1 + a$ we obtain $m = M$. According to the result available in [18] that the S_2 is a global attractor of the 3-D Lozi map (3).

4. UNBOUNDEDNESS OF SOLUTIONS OF THE 3-D LOZI MAP

In this section, we give sufficient conditions for the existence of unbounded solutions.

Theorem 4.1. If $a > 1$, $b > 0$ and $x_t, x_{t-1} > 0$, then the every orbit of the 3-D Lozi map (3) is unbounded if $x_0 > 0$.

Proof. Let $(x_t)_{t-2}$ be a solution of map (3). If $x_t > 0$ and $x_{t-1} > 0$ the 3-D Lozi map (3) can be rewritten as follows

$$x_{t+1} = 1 + bx_{t-2} + ax_t (8)$$

from (8), it follows that for all t

$$x_{t+1} = 1 + bx_{t-2} + ax_t \geq ax_t$$

by the method of iterations, we have for all integral values of t

$$x_t \geq a^t x_0$$

it is clear that the orbit is unbounded since $x_0 > 0$.

Theorem 4.2. Let $(x_t)_{t\geq-2}$ be a solution of map (3).If $a > 1$, $b < 0$ and $x_t, x_{t-1} < 0$, then the every orbit of the 3-D Lozi map (3) is unbounded if $x_0 < 0$ and t is an even number.

Proof. Let $(x_t)_{t\geq-2}$ be a solution of map (3).If $x_t < 0$ and $x_{t-1} < 0$ the 3-D Lozi map (3) can be rewritten as follows

$$x_{t+1} = 1 + bx_{t-2} - ax_t (9)$$

from (9), it follows that for all t

$$x_{t+1} = 1 + bx_{t-2} - ax_t \geq - ax_t$$

by the method of iterations, we have for all integral values of t

$$x_t \geq (-a)^t x_0$$

it is clear that the orbit is unbounded since $x_0 < 0$ and t is even.

5. Conclusion

In this letter we give the suffcient conditions for the existence of some properties of solutions in a 3-D Lozi map. that the results have been confirmed by simple analysis proof.

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A SIGN PATTERN THAT ADMITS SIGN REGULAR MATRICES OF ORDER TWO

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ABSTRACT

In this paper, conditions are identified under which a sign pattern corresponding to undirected cycles admits matrices which are sign regular of order two.

Keywords: Strictly sign regular of order *k*; sign regular of order *k*; sign pattern; closure property; interval property.

1. PRELIMINARIES

In this section, we collect several known definitions and results that will be used later on.
A matrix is called *sign regular of order k* (denoted by SSR_k) if all its minors of order *k* are non-negative or all are non-positive. It is called *strictly sign regular of order k* (denoted by SSR_k) if it is sign regular of order *k*, and all the minors of order *k* are non-zero. In other words, all minors of order *k* are non-zero and have the same sign. Such matrices are only rarely considered in the literature, see, e.g., [7], where a test for an $n \times k$ matrix with $k < n$ to be SSR_k is presented. A matrix is called *sign regular* (*SR*) if it is SR_k for all *k*, and *strictly sign regular* (*SSR*) if it is SSR_k for all *k*. Given a square matrix $A \in R^{n\times n}$ and $p \in \{1, \ldots, n\}$, consider the $\binom{n}{p}$ minors of *A* of order *p*. Each minor is defined by a set of *p* row indexes $1\leq i_1 \leq i_2 \leq \cdots \leq n$, and *p* column indexes $1\leq j_1 \leq j_2 \leq \cdots \leq n$. This minor is denoted $A(\alpha|\beta)$ where $\alpha := \{i_1, \ldots, i_p\}$and $\beta := \{i_1, \ldots, i_p\}$(with a mild abuse of notation, we will regard these sequences as sets), we suppress the curly brackets if we enumerate the indexes explicitly. We mean by k -minors of *A* all minors of *A* of order *k* and say the minors of *A* are *ssr* when they are all non-zero and have the same sign.

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2. INTRODUCTION

The most important examples of *SR* [*SSR*] matrices are totally nonnegative *TN* [totally positive *TP*]
matrices, that are, matrices with all minors nonnegative [positive]. Such matrices have applications in a number of fields including approximation theory, economics, probability theory, computer aided geometric design and other fields [3], [5], [8].

In qualitative and combinatorial matrix theory, a methodology based on the use of combinatorial information such as the signs of the elements of a matrix is very useful in the study of some properties of matrices. A matrix whose entries are chosen from the set $\{+, -, 0\}$ is called *sign pattern matrix*, the multiplicative and additive rules covering the symbols $\{+, -, 0\}$ are the same as in real numbers. A *zero* pattern is a sign pattern matrix whose entries are all equal to 0. Given an $n \times m$ real matrix $A = (a_{ij})$, we denote by *sign(A)* the sign pattern matrix obtained from *A* by replacing each one of its positive entries by + and each one of its negative entries by $-$. For an $n \times m$ sign *pattern matrix p,* we define the *sign pattern class* $C(p)$by

$$C(p) := \{A \in R^{n\times n} : sign(A) = p\}$$

A *permutation* pattern is simply a sign pattern matrix with exactly one entry in each row and column equal to $+$, and the remaining entries equal to 0. A product of the form $S^T PS$, where *S* is a square permutation pattern and *P* is a sign pattern matrix of the same order as *S*, is called a *permutation similarity*. A square sign pattern matrix whose off-diagonal entries are equal to

zero is called a *diagonal pattern*, and a product of the form DpD, where D is a diagonal pattern with no zero entries in the main diagonal and p is a sign pattern matrix of the same order as D, is called a diagonal similarity. Note that $S^T PS$ and DpD are again sign pattern matrices. The origins of sign pattern are in [9], where the author pointed to the need to solve certain problems in economics and other areas based only on the signs of entries of the matrices. The exact values of entries of the matrices may not always be known.
A sign pattern matrix p is said to *require* a certain property ρ referring to real matrices if all real matrices in $C(p)$ have the property ρ, and is said to *allow* the property ρ if some real matrices in $C(p)$ have the property ρ. In the literature, one can find, in the last few years, an increasing interest in problems that arise from the basic question of whether a certain sign pattern matrix requires (or allows) a certain property. See, e.g., [1], [4].

3. SIGN-PATTERN OF SIGN REGULAR MATRICES MATRICES OF ORDER 2 QUATIONS

In this section, we focus on the question which sign pattern matrices allow the property of belonging to the class SR_2. A graph theoretical approach will be quite useful to answer this question. Let $p = (p_{ij})$ an $n \times n$ sign pattern matrix. The graph *G(p)=(V(G),E(G)),* where the set of vertices *V(G)* is $\{1, \ldots, n\}$ and (i, j) is an edge or arc in *E(G)* if and only if $p_{ij} \neq 0$. A *path* in a graph is a sequence of edges (i_1, i_2), $(i_2, i_3), \ldots, (i_{k-1}, i_k)$ in which all vertices are distinct, except, possibly, the first and the last. The length of a path is the number of edges in the path. A *cycle* is a closed path, that is a path in which the first and the last vertices coincide. Given a cycle in (i_1, i_2), $(i_2, i_3), \ldots, (i_{k-1}, i_k), , \ldots, (i_k, i_1)$in a graph *G(p)*, where $p = (p_{ij})$ is a sign pattern matrix, we define the sign of the cycle as 1 if $p_{i_1}p_{i_2}p_{i_2}p_{i_3}, \ldots, p_{i_k}p_{i_1} = +$ and $p_{i_1}p_{i_2}p_{i_2}p_{i_3}, \ldots, p_{i_k}p_{i_1} = -$.

Remark 3.1. [1, p.2048]
If p is a sign pattern matrix whose associated graph is a directed *n*-cycle, then there is a permutation similarity that transforms p into the following form

$$p = \begin{bmatrix} 0 & p_{12} & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & p_{23} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & p_{34} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & p_{n-1,n} \\ p_{n1} & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

where $p_{n1} \neq 0$ and $p_{i,i+1} \neq 0$ for $i = 1, \ldots, n-1$. We will treat the graph as undirected when convenient. Also if p is a sign pattern matrix whose associated graph is a directed *n*-cycle with *n*-loops, i.e.,$p_{ii} \neq 0$ for all for $i = 1, \ldots, n$, then there is a permutation similarity that transforms p into the following form

$$p = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 & \ldots & 0 & 0 \\ 0 & p_{22} & p_{23} & 0 & \cdots & 0 & 0 \\ 0 & 0 & p_{33} & p_{34} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ p_{n1} & 0 & 0 & 0 & \ldots & 0 & p_{nn} \end{bmatrix}$$

Similar to Definition 3.1 [2], we introduce the following definition.

Definition 3.2.
We say that a sign pattern matrix $p = (p_{ij})$ has the loop-path property if $p_{ii}p_{i.i+1} > 0$ for every $i = 1, \dots, n$ (as a convention $p_{n,n+1} = 1$).

Theorem 3.3.
Let $p = (p_{ij})$ be an $n \times n$ sign pattern matrix with $p_{ij} \neq 0$ for all i whose associated graph *G(p)* is an undirected *n*-cycle. Then there exist a *SR₂* matrix in *C(p)* if and only if p has the loop-path property and the sign of the *n*-cycle is -1.

proof
Let $p = (p_{ij})$ be an $n \times n$ sign pattern matrix with $p_{ii} \neq 0$ for all i whose associated graph *G(p)* is an undirected *n*-cycle and there exists SR_2 matrix in $C(p)$. Without loss of generality and by Remark 4.1 we may assume that any matrix $A \in C(p)$ is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

where *sign* $(a_{ij}) = p_{ij}$ for all choices of i and j and $a_{i,i+1} \neq 0$. Suppose that there exist $i \in \{1, \dots, n-1\}$ such that $a_{ii}a_{ii+1} < 0$, (as a convention $a_{nn+1} = 1$). Since A is SR_2, we have

$$sign\ (a_{ii}a_{i+1,i+1}) = sign\ (a_{i,i+1}a_{i+1,i+2}).$$

If $a_{i+1,i+1}a_{i+1,i+2} > 0$ this contradicts *A is* SR_2. If $a_{i+1,i+1}a_{i+1,i+2} < 0$ this contradicts the fact that

$$sign\ (a_{ii}a_{i+1,i+1}) = sign\ (a_{ii}a_{i+1,i+2}).$$

Thus *p* has the loop path property and the sign of the *n*-cycle ia -1.
Conversely, assume *p* has the loop path property. *p* by permutation similarity see Remark 4.1, has the following form:

$$p = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 & \dots & 0 & 0 \\ 0 & p_{22} & p_{23} & 0 & \cdots & 0 & 0 \\ 0 & 0 & p_{33} & p_{34} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & p_{n-1,n-1} & p_{n-1,n} \\ p_{n1} & 0 & 0 & 0 & \dots & 0 & p_{nn} \end{bmatrix}$$

Let *D* be the diagonal sign pattern matrix defined by

$$D := diag(1, p_{12}, p_{12}p_{23}, p_{12}p_{23}p_{34}, \dots, p_{12}p_{23} \dots p_{1n-1n}).$$

Given that $p_{ii+1}p_{i+1,i} < 0$ for $i = 1, \dots, n-1$, it is easy to see that

$$DpD^{-1}=\begin{bmatrix} p_{11} & + & 0 & 0 & \dots & 0 & 0 \\ 0 & p_{22} & + & 0 & \cdots & 0 & 0 \\ 0 & 0 & p_{33} & + & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & p_{n-1,n-1} & + \\ p_{n1}\prod_{i=1}^{n-1} p_{i,i+1} & 0 & 0 & 0 & \dots & 0 & p_{nn} \end{bmatrix}$$

Since p has the loop-path property and the sign of n-cycle is -1, it is clear that all the 2-nontrivial minors of the matrix *B*,

$$DpD^{-1}=\begin{bmatrix} + & + & 0 & 0 & \dots & 0 & 0 \\ 0 & + & + & 0 & \cdots & 0 & 0 \\ 0 & 0 & + & + & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & + & + \\ - & 0 & 0 & 0 & \dots & 0 & + \end{bmatrix}$$

are *ssr*, i.e., *B* is a SR_2 matrix in *C(DpD*$^{-1}$*),* by using diagonal similarity, we conclude that there exists SR_2matrix in $C(p)$ which completes the proof. □

4. CONCLUSION AND FUTURE WORKS

In this work we identify conditions under which the sign pattern corresponding to undirected cycles admits SR_2 matrices. Topics for future research include sign patterns that does not correspond to undirected cycles and admit SR_2 matrices. If a sign pattern matrix $A \in SR_k$ and two arbitrary real matrices $B_1, B_2 \in C(p)$ then $sign(B_1,* B_2) \in SR_k$, we call this property the *closure property* of SR_k matrices.The question arises whether the SR_kmatrices have the closure property. Recently, we study the interval property of matrices that are strictly sign regular of given orders. To explain the interval property, we define $A^* \in R^{n\times n}$ by $A^*:=DAD$, where $D:=diag(1,-1,\dots,(-1)^{n+1})$. The transformation $*$ is usually the "checkerboard transformation." As usual, $A \leq B$ and $A < B$ for $A, B \in R^{n\times n}$ will be understood entry-wise. Let $A \leq^* B$ and $A <^* B$ if $A^* \leq B^*$ and $A^* < B^*$, respectively. The set of the matrix interval with respect to the partial ordering $\leq^*$ will be denoted by $I(R^{n\times n})$, and $[\downarrow A, \uparrow A]$ with $\downarrow A = (a_{ij}), \uparrow A = (a_{ij})$. Equivalently, a matrix interval can be represented as an interval matrix, i.e., a matrix with all entries taken from *I(R)*, the set of the compact and nonempty real intervals. We extend the properties of real matrices to matrix intervals by saying that a matrix interval has a certain property if each real matrix contained in the interval possesses this property. Matrix intervals of several classes of matrices are investigated by some mathematicians, see e.g., [6], [10]. The question arises whether a sign pattern that admits sign regular matrices of specific order have the interval property.

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QUASI-HADAMARD PRODUCT OF CERTAIN SUBCLASSES OF β-SPIRALLIKE FUNCTIONS OF ORDER α

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ABSTRACT

In this work, we obtain some results concerning the quasi-Hadamard product for subclasses $\tilde{S}T_0(\alpha,\beta)$ and $\tilde{K}T_0(\alpha,\beta)$ of β-spirallike functions of order α.

Keywords: Analytic and univalent functions; Quasi-Hadamard product.

1. INTRODUCTION AND PRELIMINARIES

Let

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_1>0,\, a_n \geq 0), \quad (1) \qquad g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_1>0,\, b_n \geq 0), \quad (2)$$

$$f_i(z) = a_{1,i} z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{1,i}>0,\, a_{n,i} \geq 0), \quad (3)$$

and

$$g_j(z) = b_{1,j} z - \sum_{n=2}^{\infty} b_{n,j} z^n \qquad (\mathrm{b}_{1,j}>0,\, \mathrm{b}_{n,j} \geq 0), \quad (4)$$

be analytic in $U = \{ z: \; |z| < 1 \}$.

A function f of the form (1) is said to be in the class $ST_0(\alpha,\beta)$ if and only if

$$\left| \frac{e^{-i\beta}}{zf'(z)/f(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \qquad (z \in U), \qquad (5)$$

for some real β and $0<\alpha<1$. Also $f \in KT_0(\alpha,\beta)$ if and only if

$$\left| \frac{e^{-i\beta}}{1+zf''(z)/f'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \qquad (z \in U), \qquad (6)$$

for some real β and $0<\alpha<1$. The classes $ST_0(\alpha,\beta)$ and $KT_0(\alpha,\beta)$ were introduced by Owa et al. [9] for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, where S is the class of all analytic and univalent in U. This class of functions has been extensively exploited in some recent articles to study subclasses of functions satisfy certain conditions(see [12-19]).

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In [9], Owa et al. proved that $f(z) \in ST_0(\alpha,\beta)$ (the class of β-spirallike functions of order α) if

$$\mathrm{Re}\left\{ e^{i\beta} \ \frac{zf'(z)}{f(z)} \right\} > \alpha, \ (7)$$

and $f(z) \in KT_0(\alpha,\beta)$ (the class of β-Robertson functions of order α) if

$$\mathrm{Re}\left\{ e^{i\beta} \ \left(1+\frac{zf''(z)}{f'(z)}\right) \right\} > \alpha, \ (8)$$

(see [1, 5]).

Using arguments, as given by Owa et al. [9], wehave the following results for classes $ST_0(\alpha,\beta)$ and $KT_0(\alpha,\beta)$.

If $f \in S$ satisfies

$$\sum_{n=2}^{\infty} \left(n + \left| n - 2\alpha e^{i\beta} \right| \right) a_n \le \left(1\text{-}\left| 1 - 2\alpha e^{i\beta} \right| \right) a_1, \qquad (9)$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta,$ then $f(z) \in ST_0(\alpha,\beta)$, and if $f \in S$ satisfies

$$\sum_{n=2}^{\infty} n \left(n + \left| n - 2\alpha e^{i\beta} \right| \right) a_n \le \left(1\text{-}\left| 1 - 2\alpha e^{i\beta} \right| \right) a_1, \qquad (10)$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta,$ then $f(z) \in KT_0(\alpha,\beta)$.

For functions defined by (1), let $\tilde{S}T_0(\alpha,\beta)$ and $\tilde{K}T_0(\alpha,\beta)$ the classes of whose coefficients satisfy the conditions (7) and (8), respectively. We note that $\tilde{S}T_0(\alpha,\beta) \subseteq ST_0(\alpha,\beta)$ and $\tilde{K}T_0(\alpha,\beta) \subseteq KT_0(\alpha,\beta)$.

We now introduce the following class of analytic functions.

Definition 1.1. A function $f(z) \in ST_m(\alpha,\beta)$ for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$ if and only if

$$\sum_{n=2}^{\infty} n^m \left(n + \left| n - 2\alpha e^{i\beta} \right| \right) a_n \le \left(1\text{-}\left| 1 - 2\alpha e^{i\beta} \right| \right) a_1. \qquad (11)$$

We note that, the class $ST_m(\alpha,\beta)$ is nonempty as the following function

$$h(z) = a_1 z - \sum_{n=2}^{\infty} \frac{\left(1\text{-}\left| 1 - 2\alpha e^{i\beta} \right| \right) a_1}{n^m \left(n + \left| n - 2\alpha e^{i\beta} \right| \right)} \lambda_n z^n, (12)$$

where $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta,$ $\mathrm{a}_1 > 0,$ $\lambda_n \ge 0$ and $\sum_{n=2}^{\infty} \lambda_n \le 1$. Accordingly, the quasi-Hadamard product of functions $c(z)$ and $d(z)$ is given by

$$c * d(z) = a_1 b_1 z - \sum_{n=2}^{\infty} a_n b_n z^n. (13)$$

(see Owa [10, 11] also, [2]-[8]).

2. MAIN RESULTS

Theorem 2.1. *For each* $i = 1,2,\cdots,p$ *and* $j = 1,2,\cdots,q$, *let the functions* $f_i(z)$ defined by (3)*be in the class* $\tilde{K}T_0(\alpha,\beta)$ *and* $g_j(z)$ defined by (4)*be in the class* $\tilde{S}T_0(\alpha,\beta)$. *Then* $f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q(z) \in ST_{2p+q-1}(\alpha,\beta)$.

Proof. Let $\partial := f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q(z)$, then

$$\partial = \left\{\prod_{i=1}^{p} a_{1,i} \prod_{j=1}^{q} b_{1,j}\right\} z - \sum_{n=2}^{\infty}\left\{\prod_{i=1}^{p} a_{n,i} \prod_{j=1}^{q} b_{n,j}\right\} z^n .$$

It sufficient to show that

$$\sum_{n=2}^{\infty}\left[n^{2p+q-1}\left(n+\left|n-2\alpha e^{i\beta}\right|\right)\left\{\prod_{i=1}^{p} a_{n,i}\prod_{j=1}^{q} b_{n,j}\right\}\right] z^n \le \left(1-\left|1-2\alpha e^{i\beta}\right|\right)\left\{\prod_{i=1}^{p} a_{1,i}\prod_{j=1}^{q} b_{1,j}\right\}.$$

Since $f_i(z) \in \tilde{K}T_0(\alpha,\beta)$, so

$$\sum_{n=2}^{\infty} n\left(n+\left|n-2\alpha e^{i\beta}\right|\right) a_{n,i} \le \left(1\text{-}\left|1-2\alpha e^{i\beta}\right|\right) a_{1,i} \qquad (i = 1,2,\cdots,p).$$

Therefore,

$$a_{n,i} \le \left[\frac{\left(1\text{-}\left|1-2\alpha e^{i\beta}\right|\right)}{n\left(n+\left|n-2\alpha e^{i\beta}\right|\right)}\right] a_{1,i}. \qquad (14)$$

Since

$$n^2\left(1\text{-}\left|1-2\alpha e^{i\beta}\right|\right) \le n\left(n+\left|n-2\alpha e^{i\beta}\right|\right),$$

it follows from (10) that

$$a_{n,i} \le n^{-2} a_{1,i} \qquad (i = 1,2,\cdots,p). \qquad (15)$$

Alsofor $g_j(z) \in \tilde{S}T_0(\alpha,\beta)$, we have

$$\sum_{n=2}^{\infty}\left(n+\left|n-2\alpha e^{i\beta}\right|\right) b_{n,j} \le \left(1\text{-}\left|1-2\alpha e^{i\beta}\right|\right) b_{1,j}\ (j = 1,2,\cdots,q). \qquad (16)$$

Hence we obtain

$b_{n,j} \le n^{-1} b_{1,j}$ $(j = 1,2,\cdots,q)$. (17)

Using (11) for $i = 1,2,\cdots,p$, (13) for $j = 1,2,\cdots,q-1$, and (12) for $j = q$, we obtain

$$\sum_{n=2}^{\infty}\left[n^{2p+q-1}\left(n+\left|n-2\alpha e^{i\beta}\right|\right)\left\{\prod_{i=1}^{p} a_{n,i}\prod_{j=1}^{q} b_{n,j}\right\}\right] z^n$$

$$\le \sum_{n=2}^{\infty}\left[n^{2p+q-1}\left(n+\left|n-2\alpha e^{i\beta}\right|\right) b_{n,q}\left\{ n^{-2p} n^{-(q-1)}\left(\prod_{i=1}^{p} a_{1,i}\prod_{j=1}^{q} b_{1,j}\right)\right\}\right]$$

$$= \left(\sum_{n=2}^{\infty}\left(n+\left|n-2\alpha e^{i\beta}\right|\right) b_{n,q}\right)\left(\prod_{i=1}^{p} a_{1,i}\prod_{j=1}^{q} b_{1,j}\right)$$

$$\le \left(1-\left|1-2\alpha e^{i\beta}\right|\right)\left\{\prod_{i=1}^{p} a_{1,i}\prod_{j=1}^{q} b_{1,j}\right\}.$$

Hence $\partial \in ST_{2p+q-1}(\alpha,\beta)$. □

Corollary 2.2. *For each* $i = 1, 2, \cdots, p$, *let the functions* $f_i(z)$ defined by (3)*belong to* $\tilde{K}T_0(\alpha, \beta)$. Then $f_1 * f_2 * \cdots * f_p \in ST_{2p-1}(\alpha, \beta)$.

Corollary 2.3. *For each* $j = 1, 2, \cdots, q$, let the functions $g_j(z)$ defined by (4) be in the class $\tilde{S}T_0(\alpha, \beta)$. Then $g_1 * g_2 * \cdots * g_q(z) \in ST_{q-1}(\alpha, \beta)$.

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COMPLEX FUZZY PARAMETERISED SOFT SET

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ABSTRACT

In this paper, we first introduce complex fuzzy parameterized soft set (CFPSS) and its related properties. We then give basic operations on CFPSS namely complement, union and intersection. Some properties of the operations are derived.

Keywords: Fuzzy soft sets; fuzzy parameterised soft sets; complex fuzzy parameterised soft set; CFPSS.

1. INTRODUCTION

In 2002, Ramot et al. introduced the innovative concept of complex fuzzy set (CFS), where the novelty lies in the range of values its membership function may attain. In contrast to traditional fuzzy membership function, this range is not limited to [0, 1], but extends to the unit circle in the complex plane. Historically, the introduction of real numbers was followed by their extension to the set of complex numbers. Thus, in this research we will sug-gest a further development from real numbers to complex numbers, which is allowed to utilize the benefits of the complex numbers and fuzzy parameterized soft set properties under our generalization concept in this research.

Initially, let us recall the development of the main concepts, which are used in this research, fuzzy set (FS), soft set (SS), fuzzy soft set (FSS), fuzzy parameterized soft set (FPSS), complex fuzzy set (CFS) and complex fuzzy soft set (CFSS). Fuzzy set contains all the possible elements in each particular context or application and vast field, where fuzzy mathematical principles are developed by extending the range of values its membership func-tion may attain from {0, 1} in classical mathematical theory to [0, 1] in fuzzy set. It was introduced by Zadeh (1965). There has been unbelievable interest in this concept due to its different applications and its ability to pro-vide solutions in many problems of control, reasoning, pattern recognition, and computer vision.

In this research we incorporate two new concepts, complex fuzzy soft set and fuzzy parameterized soft set, to introduce the innovative concept of complex fuzzy parameterized soft set. Soft set was introduced by Molodtsov (1999). It is a parameterized family of subsets of the universal set. However, to solve complicated problems in economic, engineering and environment, we cannot successfully use classical methods because of different uncertainties typical for those problems, but with soft set we can solve these problems.Later, fuzzy soft set was introduced and studied by Maji et al. (2001) and other authors like Chen et al. (2005) and Aktas et al. (2007). It is a more generalized concept, which is a combination of fuzzy set and soft set. In thedefinition of a fuzzy soft set, fuzzy subsets are used as substitutes for the crisp subsets. Hence, we can say that every (classical) soft set may be considered as a fuzzy soft set.

Fuzzy parameterized (FP) soft set was introduced by Çağman et al. (2011). He proposed a decision making method based on FP-soft set theory. Also, he illustrated an example which can be successfully applied to the problems that contain uncertainties. Besides, other researchers have applied and generalised FPSS in several fields, named but a few (Çağman et al. 2010; Bashir &Salleh 2012; Çağman& Deli 2012).

In 2011, complex fuzzy soft set (CFSS) was introduced by Nadia (2011). It is a more general concept, which is a combination of complex fuzzy set and soft set. She generalised the range of membership function of fuzzy soft set from [0, 1] to the unit circle on CFSS to introduce CFSS. She also introduced basic operations such as com-plement, union and intersection.

Çağman and Enginoglu (2010a) introduced a definition help some researchers to define the fuzzy parameter-ised soft set (FPSS) and their operations (Naim et al, 2010). More detailed theoretical study of this concept was given by Çağman and Enginoglu (2010b). The approximate function of a soft set is defined from a crisp parame-ters set to a crisp subsets of universal set. But the approximate functions of FPSS are defined from fuzzy param-eters set to the crisp subsets of the universal set.

The complex fuzzy set is characterised by a membership function, whose range is not limited to [0, 1] but ex-tend to the unit disk in the complex plane. As explained in Ramot et al. (2002) the key feature of complex fuzzy set is the presence of phase and its membership. This gives the complex fuzzy set wavelike properties that could lead in constructive and destructive interference depending on the phase value. Hence, Ramot et al. (2001) and Zhang et al. (2009) introduced several possibilities for calculating the complement, union, intersection, and other several properties for the phase term and amplitude term.

2. PRELIMANIRIES

Place In this section we recollect some relevant definitions and basic operations on fuzzy set, soft set, fuzzy soft set, complex fuzzy set, complex fuzzy soft set and fuzzyparameterisedsoft set.

Definition 2.1 (Zadeh 1965) A fuzzy set A in a universe of discourse *U* is characterised by a membership function $\mu_A(x)$ that takes values in the interval $[0,1]$.

Definition 2.2 (Ramot et al. 2001) A complex fuzzy set (CFS) *S*, defined on a universe of discourse *U*, is characterized by membership functions $\mu_S(x)$, that assign to any element $x \in U$ a complex-valued grade of membership in *S*. By definition, the values of $\mu_S(x)$, may receive all lying within the unit circle in the complex plane, and are thus of the form $\mu_S(x) = r_S(x) \cdot e^{i\,\omega_S(x)}$, where $i = \sqrt{-1}$, each of $r_S(x)$ and $\omega_S(x)$ are both real-valued, and $r_S(x) \in [0,\ 1]$.

The CFS *S* may be represented as the set of ordered pairs

$$S = \{ (x,\ \mu_S(x)) : x \in U \}.$$

Definition 2.3 (Zhang et al. 2009) Let A and B be two CFSs on *U*, and $\mu_A(x) = r_A(x) \cdot e^{i\ arg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{i\ arg_B(x)}$ their membership functions, respectively. The complex fuzzy union of A and *B*, denoted by $A \cup B$, is specified by a function

$$\mu_{A\cup B}(x) = r_{A\cup B}(x) \cdot e^{i\ arg_{A\cup B}(x)} = \max(r_A(x),\ r_B(x)) \cdot e^{i \max(arg_A(x),\ \arg_B(x))}.$$

Definition 2.4 (Zhang et al. 2009) Let A and B be two CFSs on *U*, and $\mu_A(x) = r_A(x) \cdot e^{i\ arg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{i\ arg_B(x)}$ their membership functions, respectively. The complex fuzzy intersection of A and *B*, denoted by $A \cap B$, is specified by a function

$$\mu_{A\cap B}(x) = r_{A\cap B}(x) \cdot e^{i\ arg_{A\cap B}(x)} = \min(r_A(x),\ r_B(x)) \cdot e^{i \min(arg_A(x),\ \arg_B(x))}.$$

Definition 2.5 (Zhang et al. 2009) Let$A$ be a CFS on U, and $\mu_A(x) = r_A(x)\cdot e^{i\, arg_A(x)}$ its membership function. The complex fuzzy complement of A, denoted by $\overline{A}$, is specified by a function

$\mu_{\overline{A}}(x) = r_{\overline{A}}(x)\cdot e^{i\, arg_{\overline{A}}(x)} = (1-r_A(x))\cdot e^{i(2\pi - arg_A(x))}$.

Definition 2.6(Majiet al. 2001)Let U be an initial set and E be a set of parameters. Let $F(U)$ denote the fuzzy power set of U, and let $A \subset E$. A pair is called a fuzzy soft set over U, where F is a mapping given by $F: A \to F(U)$.

Definition 2.7 (Maji et al. 2001) The union of two fuzzy soft sets (F, A) and (G, B) over a common universe U is the fuzzy soft set (H, C), where $C = A \cup B$ and $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B, \\ G(e) & \text{if } e \in B - A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$ We write $(F, A) \vee (G, B) = (H, C)$.

Definition 2.8 (Maji et al. 2001) Intersection of two fuzzy soft sets (F, A) and (G, B) over a common universe U is the fuzzy soft set (H, C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e)$ or $G(e)$ and is written as $(F, A) \wedge (G, B) = (H, C)$.

Definition 2.9 (Maji et al. 2001) The complement of a fuzzy soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F, \neg A)$, where $F^c : \neg A \to P(U)$ is a mapping given by $F^c(\alpha) = (F(\neg\alpha))^c$, $\forall \neg\alpha \in A$.

Definition 2.10 (Nadia 2010**)** Let U be an initial set and E be a set of parameters. Let $P(U)$ denote the complex fuzzy power set of U, and let $A \subset E$. A pair (F, A) is called a complex fuzzy soft set over U, where F is a mapping given by $F: A \to P(U)$.

$$F(s_j) = \{(h_k,\ r_k(x)\cdot e^{i\arg_k(x)}):\ j = \#\ \text{parameters and}\ k = \#\ \text{sets and}\ s_j \in A\}.$$

4.

Definition 2.11 (Çağman et al. 2011) Let U be an initial universe, $P(U)$ be the power set of U, E be the set of all parameters and X be a fuzzy set over E. An FP-soft set F_X on the universe U is defined by the set of ordered pairs

$F_X = \{(\mu_X(x)/x, f_X(x)) : x \in E, f_X(x) \in P(U), \mu_X(x) \in [0, 1]\}$,

where the function $f_X : E \to P(U)$ is called an approximate function such that $f_X(x) = \varnothing$ if $\mu_X(x) = 0$, and the function $\mu_X : E \to [0, 1]$ is called a membership function of FP-soft set F_X. The value of $\mu_X(x)$ is the degree of importance of the parameter x, and depends on the decision maker's requirements.

Definition 2.12(Çağman et al. 2011) Let $F_X \in FPS(U)$. The complement of F_X denoted by $F_X^{c\sim}$, is an FP-soft set defined by the approximate and membership functions as

$\mu_{X^{c\sim}}(x) = 1 - \mu_X(x)$ and $f_{X^{c\sim}}(x) = U \setminus f_X(x)$.

Definition 2.13 (Çağman et al. 2011) Let $F_X, F_Y \in FPS(U)$. The union of F_X and F_Y, denoted by $F_X \tilde{\cup} F_Y$, is defined by

$\mu_{X\tilde{\cup}Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ and $f_{X\tilde{\cup}Y}(x) = f_X(x) \cup f_Y(x)$, for all $x \in E$.

Definition 2.14(Çağman et al. 2011) Let F_X , $F_Y \in FPS(U)$. The intersection F_X and F_Y , denoted by $F_X \tilde{\cap} F_Y$, is defined by

$\mu_{X\tilde{\cap}Y}(x) = \max\{\mu_X(x),\ \mu_Y(x)\}$, and $f_{X\tilde{\cap}Y}(x) = f_X(x) \cap f_Y(x)$, for all $x \in E$.

3. COMPLEX FUZZY PARAMETRISED SOFT SET

We introduce the definition of a complex fuzzy parameterised soft set which is a generalisation of fuzzy parameterised soft set by extending the range of values of its membership function from the interval $[0,1]$ to the unite circle in the complex plane. Also, basic operations are introduced.

Formal definition

In this section, we present the formal definition of complex fuzzy parametrized soft set. Also, complex fuzzy decision set of an CFP-soft set is constructed to desine a proper decision method.

Definition 3.1.1. Let *U* be an initial universe, *P*(*U*) be the power set of *U, E* be the set of all parameters and *X* be a complex fuzzy set over *E*. A complex fuzzy parameterised soft set (CFPSS) F_X on the universe *U* is defined by the set of ordered pairs

$$F_X = \left\{\left(\mu_X(x)/x ,\ f_X(x) \right) : \forall x \in E,\ f_X(x) \in P(U),\ \mu_X(x) \in \{a : a \in \mathbb{C} \text{ and } |a| \le 1\}\right\},$$

where the function $f_X : E \to P(U)$ is called an approximate function such that $f_X(x) = \varnothing$ if $\mu_X(x) = 0 \cdot e^{i0\pi}$ and the function $\mu_X : E \to \{a \mid a \in \mathbb{C} \text{ and } |a| \le 1\}$ is called a membership function of complex FP-soft set F_X . The value of $\mu_X(x)$ is the degree of importance of the parameter *x* in periodic time and depends on the decision maker's requirements.

The difference between our complex fuzzyparameterised soft set and the previous fuzzyparameterised soft set of Çağman et al. (2011) lies in the ability to get wider range of the degree of importance of *x*, by using the properties of complex numbers.

Notes (1). Both the amplitude and phase terms may convey fuzzy information. Fuzzy information are characterized by a function from universe of discourse to [0, 1]. (Tamer et al. 2011).

(2)**.** In this research we denote the set of all CFPSS over *U* by *CFPS*(*U*).

The new concept of complex fuzzyparameterised soft set is that the sets used in the definition and example above is complex fuzzysoft sets and fuzzyparameterized soft set, characterized by complex-valued membership functions, that given by Ramot et al. in (2002), Nadia's in (2010) and Çağman et al (2011), which allows us to use the properties of complex numbers, complex fuzzysoft sets and fuzzyparameterised soft set.

We define complex fuzzy decision set of an CFP-soft set to construct a decision method by which approximate functions of a soft set are combined to produce a single complex fuzzy set that can be used to evaluate each alternative.

Definition 3.1.2. Let $F_X \in CFP(U)$. A complex fuzzy decision set of F_X , denoted by $C \sim F_X^d(s)$, is defined by

$$C \sim F_X^d(s) = \left\{ \mu_{C\sim F_X^d}(s) = r_{C\sim F_X^d}(s) e^{i2\pi\theta_{C\sim F_X^d}(s)} \big/ s \ : s \in U \right\},$$

which is a complex fuzzy set over *U*, its membership function $\mu_{C\sim F_X^d}(s)$ is defined by

$\mu_{C\sim F_X^d}(s) : U \to \{a : a \in \mathbb{C} \text{ and } |a| \le 1\}$,

$$\mu_{C\sim F_X^d}(s) = \sum_{x \in \operatorname{supp}(X)} \left(\frac{r_{C\sim F_X^d}(x)}{|\operatorname{supp}(X)|} . e^{i2\pi \frac{\theta_{C\sim F_X^d}(x)}{|\operatorname{supp}(X)|}} \right) . \chi_{f_X(x)}(s)$$

where $\displaystyle\sum_{x \in \operatorname{supp}(X)} \frac{r_{C\sim F_X^d}(x)}{|\operatorname{supp}(X)|} . e^{i2\pi \frac{\theta_{C\sim F_X^d}(x)}{|\operatorname{supp}(X)|}} = \left(\frac{1}{|\operatorname{supp}(X)|} \sum_{x \in \operatorname{supp}(X)} r_{C\sim F_X^d}(x) \right) . e^{\frac{i2\pi}{|\operatorname{supp}(X)|} \sum \theta_{C\sim F_X^d}(x)}$,

where*supp(X)* is the support set of *X*. number of importance of parameter *x*. $f_X(x)$ is the crisp subset determined by the parameter x and

$$\chi_{f_X(x)}(u) = \begin{cases} 1 & u \in f_X(x), \\ 0 & u \notin f_X(x). \end{cases}$$

4. BASIC OPERATIONS AND SOME RESULTS OF COMPLEX FUZZY PARAMETERISED SOFT SET

· In this section, we introduce the concept of complement, union and intersection of a complex fuzzyparameterised soft set by incorporating Zhang's (2009) definition for complement of complex fuzzy sets, Maji et al.'s (2001) definition for complement of fuzzy soft set and Çağman et al.'s (2011) definition for complement of fuzzy parameterised soft set.

Definition 3.2.1. Let $F_X \in CFPS(U)$. The complement F_X, denoted by F_X^c, is a CFP-soft set defined by the approximate and membership functions as

$\mu_{X^c}(x) = (1 - r_X(x)).e^{i(2\pi - 2\pi\theta(s))}$ and $f_{X^c}(x) = U \setminus f_{X^c}(x)$

Definition 3.2.2. Let $F_X, F_Y \in CFPS(U)$. The union of F_X and F_Y, denoted by $F_X \cup F_Y$, is defined by

$$F_X \cup F_Y = \left\{ \left(\mu_{X\cup Y}(x) \big/ x , \ f_{X\cup Y}(x) \right) : x \in E \right\}$$

where

$\mu_{X\cup Y}(x) = \left[\{\max(r_X(x), r_Y(x))\}.e^{i\max(\theta_X(x), \theta_Y(x))} \right]$ and $f_{X\cup Y}(x) = f_X(x) \cup f_Y(x), \ \forall x \in E.$

Definition 3.2.3. Let $F_X, F_Y \in CFPS(U)$ The intersection of F_X and F_Y, denoted by $F_X \cap F_Y$, is defined by

$$F_X \cap F_Y = \left\{ \left(\mu_{X\cap Y}(x) \big/ x , \ f_{X\cap Y}(x) \right) : x \in E \right\}$$

where

$\mu_{X\cap Y}(x) = \min(r_X(x), r_Y(x)).e^{i\min(\theta_X(x), \theta_Y(x))}$ and $f_{X\cap Y}(x) = f_X(x) \cap f_Y(x)$, for all $x \in E$.

Proposition 3.2.1. Let $F_X \in CFPS(U)$. Then $(F_X^c)^c = F_X$.

Proof : Trivial.

Proposition 3.2.2 Let F_X , F_Y , F_Z $\in$ $CFPS(U)$. Then

$F_X \cup F_X = F_X$. 2. $F_X \cup F_Y = F_Y \cup F_X$. 3. $(F_X \cup F_Y) \cup F_Z = F_X \cup (F_Y \cup F_Z)$.

$F_X \cap F_X = F_X$. 5. $F_X \cap F_Y = F_Y \cap F_X$. 6. $(F_X \cap F_Y) \cap F_Z = F_X \cap (F_Y \cap F_Z)$.

Proof: Trivial.

Proposition 3.2.3. Let F_X, F_Y $\in CFPS(U)$. Then De Morgan's Law are valid.

$$(F_X \cup F_Y)^c = F_X^c \cap F_Y^c.$$

$$(F_X \cap F_Y)^c = F_X^c \cup F_Y^c.$$

Proof: Trivial.

Proposition 3.2.4. Let $F_X, F_Y, F_Z \in CFPS(U)$. Then

$$F_X \cup (F_Y \cap F_Z) = (F_X \cup F_Y) \cap (F_X \cup F_Z).$$

$$F_X \cap (F_Y \cup F_Z) = (F_X \cap F_Y) \cup (F_X \cap F_Z).$$

Proof: Trivial.

5. Conclusion

In this research, we find out the new concept of complex fuzzy parameterised soft set, Also, we introduce the basic theoretic operations on this new concept which are, union, intersection, and complement on complex fuzzy parameterised soft set. Some propositions and relations on and between these basic theoretic operations are introduced.

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CRAMER-RAO BOUND OF DIRECTION FINDING USING MULTI-CONCENTRIC CIRCULAR ARRAYS

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ABSTRACT

Consider concentric circular arrays consisting of identical isotropic sensors. Concentric circular arrays preserve circular symmetry of the simple circular arrays, while increasing the number of spatial samples per each time instant. Direction of arrival (DOA) estimation is a key area of sensor array processing which is encountered in a broad range of important engineering applications. These applications include wireless communication, radar, sonar, among others. This paper investigates direction-finding estimation accuracy through Cramer-Rao bound derivation and analysis. It was observed that even with the same number of sensors, distributing them in a number of concentric circular arrays improves estimation accuracy.

Keywords: array signal processing, direction of arrival estimation, direction finding, Cramer-Rao bound

1. INTRODUCTION

Source direction-of-arrival (DoF) of the incoming signal from a single or multiple sources is an important technique in sensor-array signal processing[1] and refers to the problem of estimating polar-azimuth angles-of-arrival emanating from emitter(s); for example plane wave or multiple plane waves [7]. The technique is also referred to as direction finding (DF) which has been proven to play significant role in array signal processing, an important branch of signal processing with a wide range of applications especially in the world of engineering. Some of its application fields include: Sonar, radar, wireless communication, seismic systems, electronic surveillance, medical diagnosis, radio astrology, among others [1, 12].

Achievement of direction finding in signal processing makes use of elements termed as antennas or sensors either randomly distributed or arranged in the desirable geometric patterns which are either linear, planar or 3-dimensional. For instance, the already investigated sensor-array geometries in DF include uniform linear array(ULA), uniform rectangular array(URA), uniform circular array(UCA), L-shaped array, regular tetrahedral array and circular concentric array [7, 2, 13]. All these geometries are used to solve direction finding problems using different algorithms such as Maximum likelihood (ML), MUltipleSIgnal Classification (MUSIC), Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT), Cramer-Rao bound (CRB) among other techniques. For example considering a uniform circular array (UCA) geometry using CRB technique, this geometry has been investigated for direction finding in [8, 6, 3].

Importantly, each geometry aforementioned has its own advantages in DF. However, circular and concentric geometries out-weigh the other geometries based on their wide range DF advantageous allowances. Among these merits include: they offer full rotational symmetry about the origin, they are flexible in array pattern synthesis and design both in narrow band and broad band beam-forming applications, they provide almost invariant azimuth angle coverage and they can also yield invariant array pattern over a certain frequency band for beam-forming in 3-dimensions [5].

Exceptionally, circular concentric arrays have a little more advantages some of which include: they offer less mutual coupling effect due to their significant structure of the ring array [16], they yield smaller sidelobesinbeam-forming[16,14], providehigherangleresolutioncomparedtouniformcirculararraygeometries and requires less area for the same number of sensor elements [13] and they increase array's spatial aperture [15, 9, 5].

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Despite all the advantages of the circulararrays, they suffer from high side lobes in beam-forming and thus a need arises to minimize/or reduce these side-lobes. Thus the strategy of increasing the number of rings is hereby employed to reduce the effect of side lobes. Therefore, the paper considers a multi-concentric ring array that preserves circular symmetry of the simple circular array, while increasing the number of spatial samples per each time instant and offers reduced side lobes with little/or no mutual coupling in direction finding. The paper further verifies direction finding accuracy via Cramer-Rao bound derivation and analysis.
Finally, the paper is organized into six sections in which Section 1 is the introduction, Section 2 presents the statistical data model, Section 3 gives review of the Cramer-Rao bound basics, Section 4 presents the Cramer-Rao bound derivation, Section 5 presents some special cases and Section 6 gives the conclusion.

2. STATATISTICAL DATA MODEL

Consider N circular arrays with the n-th circular array of radius R_n, and containing L_n isotropic sensors uniformly arranged on the circumference for $n = 1, \cdots, N$. Let $R_n < R_{n+1}$ and $L_n < L_{n+1}$ for all n. See Figure 1.

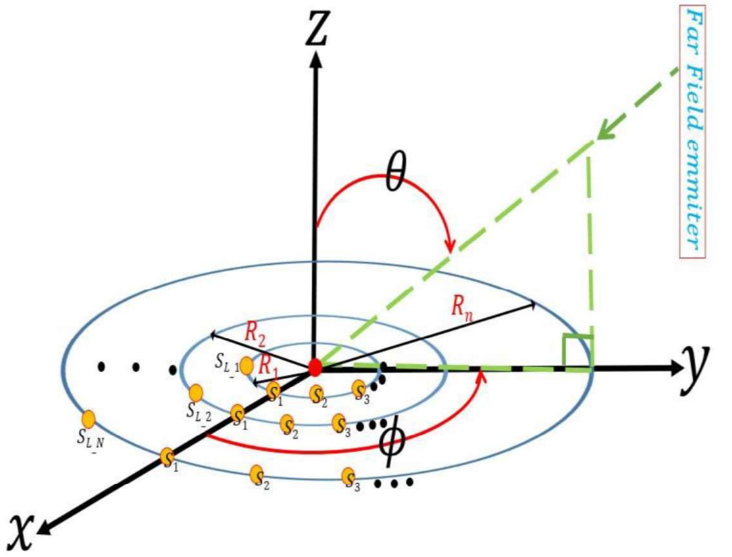

Figure 1: An N-lingth Multi-Concentric Circular Array.

The location of the ℓ^{th} sensor on the R_n^{th} radius circular array is given by

$$\boldsymbol{p}_\ell = \left[R_n \cos\left(\frac{2\pi(\ell_n - 1)}{L_n}\right), R_n \sin\left(\frac{2\pi(\ell_n - 1)}{L_n}\right), 0\right]^{\boldsymbol{T}} \tag{1}$$

for $\ell_n = 1, 2, 3, \cdots, L_n$

Let $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ be the elevation and azimuth angles, respectively, of a source with an incident wavelength λ. Then, the array mainifold vector is given as

$$\boldsymbol{a}(\theta, \phi) = \begin{bmatrix} \boldsymbol{a_1}(\theta, \phi) \\ \boldsymbol{a_2}(\theta, \phi) \\ \vdots \\ \boldsymbol{a}_N(\theta, \phi) \end{bmatrix} \tag{2}$$

where

$$[\boldsymbol{a}_n(\theta, \phi)]_{\ell_n} = \exp\left\{j \frac{2\pi R_n}{\lambda} \sin(\theta) \cos\left(\phi - \frac{2\pi(\ell_n - 1)}{L_n}\right)\right\}. \tag{3}$$

Consider a collected dataset $\{z(m), m = 1,2,3,\cdots,M\}$, where m is the time index and

$$\boldsymbol{z}(m) = \boldsymbol{a}(\theta,\phi)s(m) + \boldsymbol{n}(m) \tag{4}$$

is a $\sum_{n=1}^{N} L_n \times 1$ and $\boldsymbol{n}(m)$ is a $\sum_{n=1}^{N} L_n \times 1$ vector modelled as a complex-valulued zero-mean additive white Gaussian noise (AWGN) with a prior known variance of σ_n^2 and $s(m)$ is a scalar incident signal modelled as a white Gaussian complex-value random sequence with a prior known variance of σ_s^2. The noise,$\{\boldsymbol{n}(m), \forall m\}$is white both across time (m) and across space (i.e across the components in the$\sum_{n=1}^{N} L_n$ elements of each vector $\boldsymbol{n}(m)$).

3. REVIEW OF CRAMER-RAO BOUND BASICS

Let

$$\check{\boldsymbol{z}} := [\{\boldsymbol{z}(1)\}^T, \{\boldsymbol{z}(2)\}^T, \cdots, \{\boldsymbol{z}(M)\}^T]^T = \boldsymbol{s} \otimes \boldsymbol{a}(\theta,\phi) + \check{\boldsymbol{n}} \tag{5}$$

be the dataset representing M number of discrete-time samples. In Eq. (5), superscript $^{\mathbf{T}}$denotes transposition, $\otimes$ denotes the Kronecker product and

$$s := [s(1), s(2), \cdots, s(M)]^T,$$
$$\check{\boldsymbol{n}} := [\{\boldsymbol{n}(1)\}^T, \{\boldsymbol{n}(2)\}^T, \cdots, \{\boldsymbol{n}(M)\}^T]^T.$$

Collect the two to-be-estimated scalar parameters as entries of the 2×1 vector $\xi := [\theta, \phi]$. The fisher information matrix (FIM), $\boldsymbol{F}(\boldsymbol{\xi})$ has a (k,r)-th entry equal to (see Eq. (3.8) on page 72 of [4])

$$[\boldsymbol{F}(\boldsymbol{\xi})]_{k,r} = 2\mathrm{Re}\left\{\left[\frac{\partial\boldsymbol{\mu}}{\partial\xi_k}\right]^H \boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\xi_r}\right\} + \mathrm{Tr}\left\{\boldsymbol{\Gamma}^{-1}\left[\frac{\partial\boldsymbol{\Gamma}}{\partial\xi_k}\right]^H \boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\Gamma}}{\partial\xi_r}\right\}, \tag{6}$$

where Re$\{\cdot\}$ signifies the real-value part of the entity inside the curly brackets, Tr$\{\cdot\}$ denotes the trace of the entity inside the curly brackets, the superscript ^{H}indicates conjugate transposition.

For the data's statistical model,

$$\boldsymbol{\mu} := E[\check{\boldsymbol{z}}] = \boldsymbol{s} \otimes \boldsymbol{a}(\theta,\phi) \tag{7}$$

$$\boldsymbol{\Gamma} := E[(\check{\boldsymbol{z}} - \boldsymbol{\mu})(\check{\boldsymbol{z}} - \boldsymbol{\mu})^{\boldsymbol{H}}] = \sigma_n^2 \boldsymbol{I}_{M\sum_{n=1}^{N} L_n} \tag{8}$$

where $E[\cdot]$ represents the statistical expectation of the entity inside the square brackets and $\boldsymbol{I}_{M\sum_{n=1}^{N} L_n}$symbolizes an identity matrix of size $M\sum_{n=1}^{N} L_n$.

Because $\boldsymbol{\Gamma}$ is functionally *in*dependent of both $\theta$and ϕ, as shown in Eq. (8), the second term of Eq. (6) equals zero. Eq. (6) may be simplified to

$$[\boldsymbol{F}(\boldsymbol{\xi})]_{k,r} = \frac{2}{\sigma_s^2}\mathrm{Re}\left\{\left[\frac{\partial\boldsymbol{\mu}}{\partial\xi_k}\right]^H \frac{\partial\boldsymbol{\mu}}{\partial\xi_r}\right\},$$

where

$$\left[\frac{\partial\boldsymbol{\mu}}{\partial\boldsymbol{\xi_k}}\right]^{\boldsymbol{H}} \frac{\partial\boldsymbol{\mu}}{\partial\boldsymbol{\xi_r}} = \left[\boldsymbol{s} \otimes \frac{\partial\boldsymbol{a}(\theta,\phi)}{\partial\boldsymbol{\xi_k}}\right]^{\boldsymbol{H}}\left[\boldsymbol{s} \otimes \frac{\partial\boldsymbol{a}(\theta,\phi)}{\partial\boldsymbol{\xi_r}}\right]$$
$$= \underbrace{\boldsymbol{s}^{\boldsymbol{H}}\boldsymbol{s}}_{:=M\sigma_s^2} \otimes \left\{\left[\frac{\partial\boldsymbol{a}(\theta,\phi)}{\partial\boldsymbol{\xi_k}}\right]^{\boldsymbol{H}}\left[\frac{\partial\boldsymbol{a}(\theta,\phi)}{\partial\boldsymbol{\xi_r}}\right]\right\}$$
$$= M\sigma_s^2\left[\frac{\partial\boldsymbol{a}(\theta,\phi)}{\partial\boldsymbol{\xi_k}}\right]^H\left[\frac{\partial\boldsymbol{a}(\theta,\phi)}{\partial\boldsymbol{\xi_r}}\right].$$

1. **Hence**

$$[\boldsymbol{F}(\boldsymbol{\xi})]_{k,r} = 2\mathbf{M}\frac{\sigma_s^2}{\sigma_n^2}\mathrm{Re}\left\{\left[\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \xi_{\boldsymbol{k}}}\right]^{\boldsymbol{H}}\left[\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \xi_{\boldsymbol{r}}}\right]\right\}. \tag{9}$$

The Fisher information matrix equals

$$\boldsymbol{F}(\boldsymbol{\xi}) = \begin{bmatrix} F_{\theta,\theta} & F_{\theta,\phi} \\ F_{\phi,\theta} & F_{\phi,\phi} \end{bmatrix}, \tag{10}$$

the inverse of which gives Cramer-Rao bound ofθ and ϕ:

$$\begin{bmatrix} \mathrm{CRB}(\theta) & * \\ * & \mathrm{CRB}(\phi) \end{bmatrix} = \begin{bmatrix} F_{\theta,\theta} & F_{\theta,\phi} \\ F_{\phi,\theta} & F_{\phi,\phi} \end{bmatrix}.^{-1} \tag{11}$$

The Cramer-Rao Bound Derivation

From Eq. (2), we have

$$\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \xi_{\boldsymbol{k}}} = \left[\left[\frac{\partial \boldsymbol{a_1}(\theta,\phi)}{\partial \xi_{\boldsymbol{k}}}\right]^{\boldsymbol{H}}, \left[\frac{\partial \boldsymbol{a_2}(\theta,\phi)}{\partial \xi_{\boldsymbol{k}}}\right]^{\boldsymbol{H}}, \cdots, \left[\frac{\partial \boldsymbol{a_N}(\theta,\phi)}{\partial \xi_{\boldsymbol{k}}}\right]^{\boldsymbol{H}}\right]^{\boldsymbol{T}}, \tag{12}$$

where

$$\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \theta} = j\frac{2\pi}{\lambda}R_n\cos(\theta)\begin{bmatrix} \cos(\phi) \\ \cos\left(\phi - \frac{2\pi}{L_n}\right) \\ \vdots \\ \cos\left(\phi - \frac{2\pi(\ell_n - 1)}{L_n}\right) \end{bmatrix} \odot \boldsymbol{a_n}(\theta,\phi), \tag{13}$$

$$\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \phi} = -j\frac{2\pi}{\lambda}R_n\sin(\theta)\begin{bmatrix} \sin(\phi) \\ \sin\left(\phi - \frac{2\pi}{L_n}\right) \\ \vdots \\ \sin\left(\phi - \frac{2\pi(\ell_n - 1)}{L_n}\right) \end{bmatrix} \odot \boldsymbol{a_n}(\theta,\phi). \tag{14}$$

In Eq. (13) and Eq. (14), $\odot$denotes the Hadamard product.

From Eq. (12)-Eq. (14):

$$\left[\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \theta}\right]^{\boldsymbol{H}}\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \theta} = \sum_{n=1}^{N}\left\{\left[\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \theta}\right]^{\boldsymbol{H}}\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \theta}\right\}$$
$$= \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)^2\cos^2(\theta)\textstyle\sum_{n=1}^{N}L_nR_n^2, \tag{15}$$

$$\left[\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \phi}\right]^{\boldsymbol{H}}\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \theta} = \sum_{n=1}^{N}\left\{\left[\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \theta}\right]^{\boldsymbol{H}}\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \phi}\right\}$$
$$= 0, \tag{16}$$

$$\left[\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \phi}\right]^{\boldsymbol{H}}\frac{\partial \boldsymbol{a}(\theta,\phi)}{\partial \phi} = \sum_{n=1}^{N}\left\{\left[\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \phi}\right]^{\boldsymbol{H}}\frac{\partial \boldsymbol{a_n}(\theta,\phi)}{\partial \phi}\right\}$$
$$= \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)^2\sin^2(\theta)\textstyle\sum_{n=1}^{N}L_nR_n^2\,. \tag{17}$$

Using Eq. (15)- Eq. (17) in Eq. (9), we have

$$F_{\theta,\theta} = M\left(\frac{2\pi}{\lambda}\right)^2\left(\frac{\sigma_s}{\sigma_n}\right)^2\cos^2(\theta)\sum_{n=1}^{N} L_n R_n^2, \quad (18)$$

$$F_{\theta,\phi} = 0, \quad (19)$$

$$F_{\phi,\phi} = M\left(\frac{2\pi}{\lambda}\right)^2\left(\frac{\sigma_s}{\sigma_n}\right)^2\sin^2(\theta)\sum_{n=1}^{N} L_n R_n^2. \quad (20)$$

Using Eq. (18)- Eq. (20) in Eq. (11), we have

$$\mathrm{CRB}(\theta) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\sec^2(\theta)}{\sum_{n=1}^{N} L_n R_n^2}, \quad (21)$$

$$\mathrm{CRB}(\phi) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\csc^2(\theta)}{\sum_{n=1}^{N} L_n R_n^2}, \quad (22)$$

Special Cases

A Single Circular Array

From Eq. (21)- Eq. (22):

$$\mathrm{CRB}(\theta) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\sec^2(\theta)}{L_1 R_1^2}, \quad (23)$$

$$\mathrm{CRB}(\phi) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\csc^2(\theta)}{L_1 R_1^2}. \quad (24)$$

These results agree with the results obtained in [8, 6, 3, 11].

A 2-Circle Array

From Eq. (21)- Eq. (22):

$$\mathrm{CRB}(\theta) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\sec^2(\theta)}{L_1 R_1^2 + L_2 R_2^2}, \quad (25)$$

$$\mathrm{CRB}(\phi) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\csc^2(\theta)}{L_1 R_1^2 + L_2 R_2^2}. \quad (26)$$

These results agree with the results obtained in [11].

Equal Angular Spacing

To maintain equal angular spacing between any two consecutive sensors in each circular array, let $R_n = nR_1$ and $L_n = nL_1$. Then fromEq. (21)- Eq. (22):

$$\mathrm{CRB}(\theta) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\sec^2(\theta)}{L_1 R_1^2 \sum_{n=1}^{N} n^3}, \quad (27)$$

$$\mathrm{CRB}(\phi) = \frac{1}{M}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{\sigma_n}{\sigma_s}\right)^2\frac{\csc^2(\theta)}{L_1 R_1^2 \sum_{n=1}^{N} n^3}. \quad (28)$$

Eq. (27)- Eq. (28) can be re-expressed as

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\mathrm{CRB}(\theta) = \frac{1}{4\pi^2}\frac{\sec^2(\theta)}{L_1 R_1^2 \sum_{n=1}^{N} n^3}, \quad (29)$$

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\mathrm{CRB}(\phi) = \frac{1}{4\pi^2}\frac{\csc^2(\theta)}{L_1 R_1^2 \sum_{n=1}^{N} n^3}, \quad (30)$$

Now, to compare signal's direction of arrival estimation accuracy using different number of rings for the concentric circular arrays, we will consider an equal number of sensors. As an example, let the total number of sensors be considered be 60. In addition, let $R_1 = 1$.

A Single Circular Array

From Eq. (29)- Eq. (30) and using $L_1 = 60$ and $R_1 = 1$, we have

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\theta) = \frac{\sec^2(\theta)}{4\pi^2\times 60}, \tag{31}$$

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\phi) = \frac{\csc^2(\theta)}{4\pi^2\times 60}, \tag{32}$$

A 2-Circle Array

From Eq. (29)- Eq. (30) and using $L_1 = 20$ and $R_1 = 1$, we have

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\theta) = \frac{\sec^2(\theta)}{4\pi^2\times 180}, \tag{33}$$

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\phi) = \frac{\csc^2(\theta)}{4\pi^2\times 180}, \tag{34}$$

Eq. (33)- Eq. (34) corresponds to $L_1 = 20, L_2 = 40\ , R_1 = 1$ and $R_2 = 2$.

A 3-Circle Array

From Eq. (29)- Eq. (30) and using $L_1 = 10$ and $R_1 = 1$, we have

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\theta) = \frac{\sec^2(\theta)}{4\pi^2\times 360}, \tag{35}$$

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\phi) = \frac{\csc^2(\theta)}{4\pi^2\times 360}, \tag{36}$$

Eq. (35)- Eq. (36) corresponds to $L_1 = 10, L_2 = 20\ , L_3 = 30, R_1 = 1, R_2 = 2,$ and $R_3 = 3$.

A 4-Circle Array

From Eq. (29)- Eq. (30) and using $L_1 = 6$ and $R_1 = 1$, we have

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\theta) = \frac{\sec^2(\theta)}{4\pi^2\times 600}, \tag{37}$$

$$M\left(\frac{1}{\lambda}\frac{\sigma_s}{\sigma_n}\right)\text{CRB}(\phi) = \frac{\csc^2(\theta)}{4\pi^2\times 600}, \tag{38}$$

Eq. (37)- Eq. (38) corresponds to $L_1 = 6, L_2 = 12\ , L_3 = 18, L_4 = 24, R_1 = 1, R_2 = 2,, R_3 = 3$ and $R_4 = 4$.

It is clear from Eq. (31)- Eq. (38), that even with the same number of sensors, distributing them in a number of concentric circular arrays improves estimation accuracy.

4. CONCLUSION

A multiple number of concentric circular sensor array grid referred here as multi-concentric circular array has been proposed. The direction-of-arrival estimation accuracy using such a multi-concentric circular array grid has been analytically determined through Cramer-Rao bound derivation. It has been observed that the Cramer-Rao bound decreases with increase in the number of concentric arrays while maintaining the same number of sensors. This observation would help direction finders to economically utilize a given number of sensors.

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A NEW HYBRID CONJUGATE GRADIENT METHOD FOR UNCONSTRAINED OPTIMIZATION UNDER EXACT LINE SEARCH[1]

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ABSTRACT

In this paper, based on some famous previous conjugate gradient methods, a new hybrid conjugate gradient coefficient was proposed for unconstrained optimization. The proposed parameter β_k^{HTM} is computed as a combination of β_k^{HS} (Hestenes-Steifel formula), β_k^{LS} (Liu –storey formula) and β_k^{RMIL} (Rivaie formula) to exploit attractive features of each. The algorithm uses the exact line search. Numerical results and their performance profiles are reported which show that the proposed method is promising. The numerical results also have shown that the new formula for β_k performs far better than the original Hestenes-Steifel, Liu –storey and the Rivaie methods.

Keywords: Hybrid conjugate gradient method; exact line search; unconstrained optimization.

1. INTRODUCTION

Conjugate gradient methods (CGMs for short) are very efficient for solving large-scale unconstrained optimization problem, especially when the dimension n is large. CGMs have been mainly designed for solving problems in the following form:

$$\min f(x)\,, \quad x \in R^n \tag{1}$$

where $f: R^n \to R$ Is continuously differentiable function, the form of iterative method to solve unconstrained optimization problem is given by

$$x_{k+1} = x_k + \alpha_k d_k \quad k= 0, 1, 2 \tag{2}$$

Where x_k is the current iterate, α_k is the positive step size achieved by carrying out a one dimensional search, known as the 'line searches'. The most common is the exact line search which is

$$f(x_k + \alpha_k d_k) = \min_{\alpha \geq 0} f(x_k + \alpha\, d_k) \tag{3}$$

and d_k is the search direction defined by

$$d_k = \begin{cases} -g_k\,, & if\ k = 0\,; \\ -g_k \;+\; \beta_k d_{k-1}\,, & if\ k \geq 1 \quad, \end{cases} \tag{4}$$

where β_k a parameter and g_k is the gradient of $f(x)$ at x_k.

In the linear CGMs or nonlinear CGMs the parameter β_k is called conjugate gradient coefficient [27]. Different choices of β_k will yield different CG method. Table 1 arranges a sequential list of some choices for the well-knwon CG parameter.

Table 1. Various choices for the classical CG parameter

$\beta_k^{HS} = \frac{g_k^T (g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}}$	(Hestenses –Stiefel [13], 1952)	(5)
$\beta_k^{FR} = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}}$	(Fletcher –Reeves [11], 1964)	(6)
$\beta_k^{PRP} = \frac{g_k^T (g_k - g_{k-1})}{g_{k-1}^T g_{k-1}}$	(Polak-Ribiere –Polyak [21, 22], 1969)	(7)
$\beta_k^{CD} = \frac{g_k^T g_k}{d_{k-1}^T g_{k-1}}$	(Conjugate Descent [10], 1987)	(8)
$\beta_k^{LS} = -\frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T g_{k-1}}$	(Liu –storey [19], 1991)	(9)
$\beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}}$ (10)	(Dai –Yuan, [6], 1999)	

There are frequent research on convergence properties of these methods (see Zoutendijk [27], Powell [23], Z.Wei [25], Zhi- Feng Dai [5], Al-Baali [2], Min Li [18] and Dai and Yuan [7]).

For non-quadratic objective functions, the global convergence property of FR method was proved [11, 27], when Strong Wolfe line search was used. The PRP method has no global convergence under some traditional line searches. Some convergent versions were proposed by using some new complicated line searches, or through restricting the parameter to a nonnegative number [18]. The CD method and DY method were proved to have global convergence under Strong Wolfe line search [5, 25]. However, to the best of our knowledge, the global convergence of PRP, LS and HS methods have not been established under all mentioned line searches. The main reason is that many CGMs cannot guarantee the descent of objective function values at each iterative.

In the latest years, based on the above formulas and their hybridization, many works putting effort into seeking for new CGMs with not only good convergence property but also excellent numerical effect were published. Nazareth [20] regarded the FR, PRP, HS, and DY formula as the four leading contenders for the scalar β_k and proposed two parameter family of conjugate gradient method. Wei et al [25], proposed a variation of the FR method which is called the VFR method.Hai Huangm, et al [16] modified LS,Zhi- Feng Dai [5] modified HS and Zhang extended the result of the HS [17] method and proposed the NHS method.Another famous CG method is the RMIL method, denoted by the name of the researchers: Rivaie, Mustafa, Ismail and Leong [24]. Its CG coefficient is written as

$$\beta_k^{RMIL} = \frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T d_{k-1}} \tag{11}$$

Some well-known CGMs have strong convergence property like FR, DY, and CD, but they may not perform well. Others like PRP, HS, and LS may not converge but they perform well. So hybrid CGMs has been devised to use and combine the attractive features of the well-known conjugate gradient algorithms.This reason leads Powell [23] to modify the PRP method.By the same motivations, Touati-Ahmed and Storey [1] extend AL-Baali's [2] convergence result on the FR method.DY,Dai and Yuan [7] propose a family of globally convergent conjugate methods. A new hybrid CG is considered by Djordjević [8] wehre the conjugate gradient parameter β_k is computed as a convex combination of β_k^{CD} and β_k^{LS}.Hu and Storey [15] suggest the formula

$$\beta_k^{HHUS} = max\left\{0\ , min\ \{\beta_k^{PRP}\ , \beta_k^{FR}\}\right\} \tag{13}$$

Gilbert and Nocedal [12] extend (13) and propose the formula

$$\beta_k^{HGN} = max\left\{-\beta_k^{FR}\ , min\ \{\beta_k^{PRP}\ , \beta_k^{FR}\}\right\} \tag{14}$$

Recently Xiao Xu and Fan-yu Kong [26] make a linear combination with parameters β_k of the DY method and the HS method.More recently Yasir [28] proposed a new hybrid CG similar to WYL.

2. NEW HYBRID CG METHOD

During the last years, much massive conducted effort has been committed to develop new modifications of CGMs, as we mention before, which do not only possess strong convergence properties, but they are also computationally superior to the classical methods. As result to that hundreds of variants Conjugate Gradient algorithms have been confirmed. A survey including 40 nonlinear Conjugate Gradient algorithms for unconstrained optimization is given by Andrei [4].

In this section, enlightened by above-mentioned ideas [12, 13], we suggest our β_k which named as $\beta_k^{HTM^*}$. Where HTM^*represents Hybrid Tala't and Mustafa.

$$\beta_k^{HTM^*} = max\left\{\beta_k^{RMIL}, min\left\{\beta_k^{LS}, \beta_k^{HS}\right\}\right\} \quad (15)$$

The algorithm is given as follows:

Algorithm 1

Step 1: Initialization. Given $x_0 \in R^n, \varepsilon \geq 0$, set $d_0 = -g_0$ if $\|g_0\| \leq \varepsilon$ then stop.
Step 2: Compute α_k byEq. (3).
Step 3: Let $x_{k+1} = x_k + \alpha_k d_k$, $g_{k+1} = g(x_{k+1})$ if $\|g_{k+1}\| \leq \varepsilon$ then stop.
Step 4: Compute β_k by (15), and generate d_{k+1} byEq. (4).
Step 5: set $k = k + 1$ go to Step 2.

Global convergence properties

In this section, the convergent properties of $\beta_k^{HTM^*}$ will be studied. We only show the result of convergence for common CG method. To verify the convergence, we assumed that every search direction d_k should fulfill the descent condition

$$g_k^T d_k < 0 \quad (16)$$

for all $k > 0$.

If there exist a constant $\lambda > 0$ for all $k > 0$ then, the search directions satisfy the following sufficient descent condition

$$g_k^T d_k \leq -\lambda \|g_k\|^2 \quad (17)$$

The following Theorem is very essential in establishing sufficient descent condition.

Theorem: *Consider a CG method with the search direction (4) and $\beta_k^{HTM^*}$ given as (15) then condition (17) holds for all $k > 0$.*

Proof. If $k = 0$ then it is clear that $g_0^T d_0 = -\lambda \|g_k\|^2$. Hence, condition (17) holds true. We also need to show that for$k \geq 1$, condition (17) will also hold true.
From (4), multiply both sides by g_{k+1}^T , we obtain

$$g_{k+1}^T d_{k+1} = g_{k+1}^T(-g_{k+1} + \beta_{k+1} d_k)$$
$$= -\|g_{k+1}\|^2 + \beta_{k+1} g_{k+1}^T d_k$$

For exact line search, we know that $g_{k+1}^T d_k = 0$. Thus,

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2$$

Therefore, it implies that d_{k+1} is a sufficient descent direction. Hence,

$$g_k^T d_k \leq -\lambda \|g_k\|^2$$

holds true. The proof is completed□.

3. NUMERICAL RESULT AND DISCUSSION

In order to check the efficiency of HTM^*, we compare HTM^* method with all classical methods . Table 2 shows the computational performance of R2015a MATLAB program on a set of unconstrained optimization test problems. We select randomly 25 test functions from Andrei [3]

In this test, we choose $\varepsilon = 10^{-6}$ and stopping criteria is set to $\|g_k\| \leq \varepsilon$ as Hillstron [14] recommended. Three initial points are chosen starting from a point closer to the solution point to a point far away from the solution point, so that it can be used to test the global convergence of the new CG coefficient. The dimensions n of 25 problems are 2, 4, 10, 100,500 and 1000.

In some cases, the calculations blocked due to the failure of the line search to find the positive step size, and thus it was considered as a fail. Numerical results are compared comparative to the number of iteration (NOI) and CPU time. We use the performance profile presented by Dolan and More [9] to get the performance results that shown in Figure 1, Figure 2, Figure 3 and Figure 4.

The CPU processor used was Intel (R) Core TM i3-M350 (2.27GHz), with RAM 4 GB.

Table 2. List of Problem Functions

NO	Function	Dim	Initial point
1	SIX HUMP CAMEL	2	(-1,-1), (3,3), (50,50)
2	TRECCANI	2	(0.5,0.5),(15,15),(150,150)
3	ZETTL	2	(-2,-2),(0.3,0.3),(5,5)
4	QUARTIC	4	(10,..,10),(50,..,50),(100,..,100)
5	EXTENDED HIMMELBLAU	4	(-4,..,-4),(-1.5,..,-1.5),(1,..,1)
6	EXTENDED MARTOS	10	(-2,..,-2),(0.5,..,0.5),(2,..,2)
7	QUADRATIC QF2	100,500,1000	(1,..,1),(15,..,15),(60,..,60)
8	GENERALZED QUARTIC	100,500,1000	(-0.5,..,-0.5),(1,..,1),(6,..,6)
9	WHITE AND HOLST	100,500,1000	(-2,..,-2),(2,..,2),(9,..,9)
10	FLETCHCR	100,500,1000	(-4,..,-4),(3,..,3),(11,..,11)
11	ROSENBROCK	100,500,1000	(5,..,5),(25,..,25),(30,..,30)
12	EXTENDED DENSCHNB	100,500,1000	(1,..,1),(16,..,16),(25,..,25)
13	EXTENDED BEALE	100,500,1000	(0.5,..,0.5),(2,..,2),(11,..,11)
14	EXTENDED TRIDIAGONAL	100,500,1000	(3,..,3),(9,..,9),(50,..,50)
15	DIAGONAL4	100,500,1000	(0.2,..,0.2),(60,..,60),(200,..,200)
16	SUM SQUARES	100,500,1000	(-1,..,-1),(60,..,60),(150,..,150)
17	SHALOW	100,500,1000	(0.2,..,0.2),(3,..,3),(30,..,30)
18	PERTURBD QUADRATIC	100,500,1000	(0.5,..,0.5),(2,..,2),(12,..,12)
19	DIXON AND PRICE	100,500,1000	(0.2,..,0.2),(0.4,..,0.4),(16,..,16)
20	QUADRATIC QF1	100,500,1000	(1.5,..,1.5),(5,..,5),(20,..,20)
21	NONDIA	100,500,1000	(3,..,3),(7.5,..,7.5),(50,..,50)
22	DQDRTIC	100,500,1000	(10,..,10),(60,..,60),(100,..,100)
23	SINQUAD	100,500,1000	(4,..,4),(20,..,20),(60,..,60)
24	GENERALIZED QUARTIC GQ2	100,500,1000	(0.5,..,0.5),(15,..,15),(25,..,25)
25	EXTENDED QUADRATIC PENALTY QP2	100,500,1000	(1,..,1),(10,..,10),(50,..,50)

In [9] Dolan and More offered a model to evaluate and compare the performance of the set solvers S on a test set P. Assuming n_s solvers and n_p problems exists, for each problem p and solver s, they defined

$t_{p,s}$ = computing time (NOI. or CPU time) required to solve problems$p$ by solver s.

Wanting a standard form for evaluations, they compared the performance of problem p by solver s with the best performance for any solver to the same problem using the performance ratio

$$r_{p,s} = \frac{1}{min\{t_{p,s} : \mathrm{s} \in \mathrm{S}\}}$$

Assume that a parameter $r_M \geq r_{p,s} \forall\, p, s$ is selected, and $r_M = r_{p,s}$ if and only if solver s does not solve problem$p$. The performance of solver s on any given problem might be of concern, but because we would like to achievement an overall valuation of the performance of the solver, then it was defined

$$p_s(t) = \frac{1}{n_p}\{p \in P: r_{p,s} \leq t\}$$

Thus $p_s(t)$is the possibility for solver s ∈ S that a performance ratio $r_{p,s}$ was within a factor t ∈ R of the best possible ratio. Then, functionp_s is the cumulative distribution function for the performance ratio. The performance profile $p_s: R \rightarrow [0,1]$for a solver was a non-decreasing, piecewise, and continuous from the right. The value of $p_s(1)$is the possibility that the solver will earn over the rest of the solvers. In general, a solver with high values of $P(\tau)$or at the top right of the figure is superior or signify the best solver.

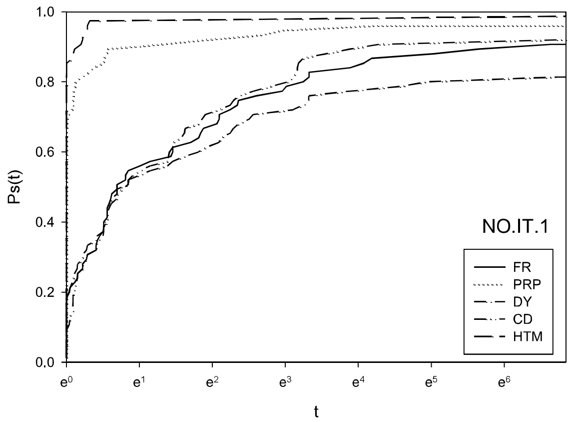

Figure 1: Performance profile based on NOI

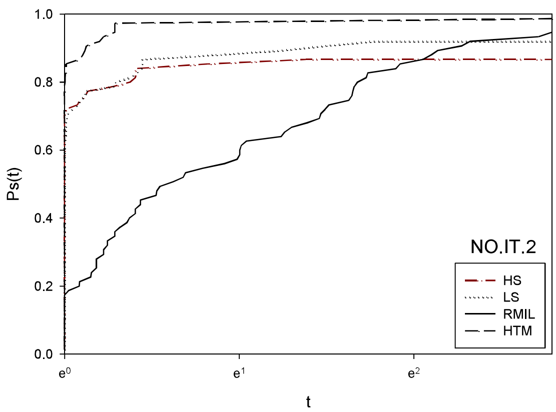

Figure 2: Performance profile based on NOI

.

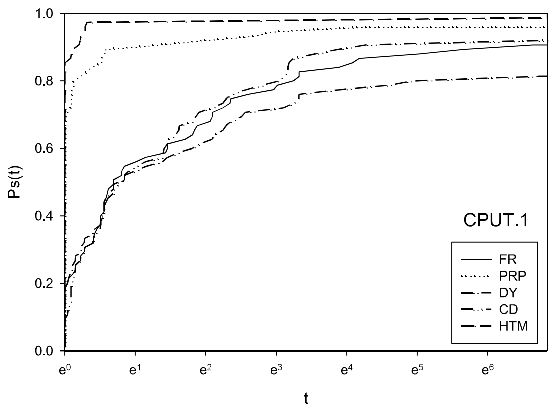

Figure. 3: Performance profile based on CPU time

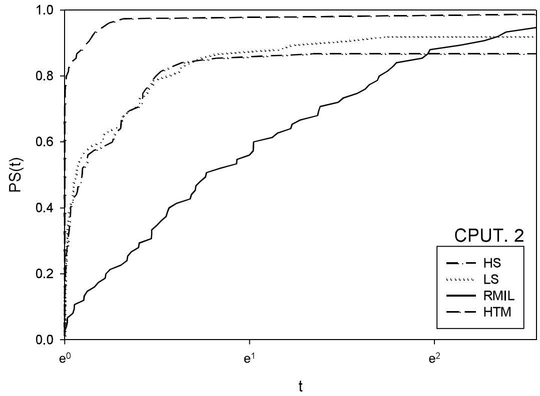

Figure. 4: Performance profile based on CPU time

Figures show the performance profile of all methods we used based on NOI and CPU time.All figures illustrate that HTM^* perform better than the other methods, since it can solve almost all of the test problems and reach 99% percentage. Comparing with DY, FR, CD, PRP, HS, LS and RMIL that don't exceed 81%, 90%, 92%, 96%, 86%, 92%, 94% respectively in solving the given test problems. To sum up, our numerical results propose a new efficient conjugate gradient method.

CONCLUSION

In this paper, the resecher have studied a new hybrid method for solving unconstrained optimization. Hedisplayed that the new method fulfills the sufficient descent condition under exact line search. The outcome of the numerical tests shows that the given method is modest when compared to other CGMs. In future, testing this new method under different search rules is recommended.

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ON SECOND ORDER PERTURBED STATE-DEPENDENT SWEEPING PROCESS

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ABSTRACT

Using a discretization approach, the existence of solutions for a class of second order differential inclusion is stated. The right hand side of the problem is governed by the so-called nonconvex state-dependent sweeping process and contains an unbounded perturbation, that is the external forces applied on the system. Thanks to some recent concepts of set's regularity and nonsmooth analysis, we extend existence results for nonconvex equi-uniformly subsmooth sets. The construction is based on the Moreau's catching-up algorithm. We give an application to the antiplane frictional contact problem, where the friction is modeled by Tresca's law.

Keywords: Differential inclusion; nonconvex sweeping process; subsmooth sets; unbounded perturbation

1. INTRODUCTION

The perturbed second order state-dependent nonconvex sweeping process is an evolution differential inclusion governed by the normal cone to a mobile set depending on both time and state variables, of the following form:

$$(P)\begin{cases} -\dot{u}(t) \in N_{Q(t,v(t))}\big(u(t)\big) + F\big(t,u(t),v(t)\big), & a.e.\ t \in [0,T];\\ v(t) = b + \int_0^t u(s)ds, & \forall t \in [0,T];\\ u(t) = a + \int_0^t \dot{u}(s)ds, & \forall t \in [0,T];\\ u(t) \in Q(t,v(t)), & \forall t \in [0,T], \end{cases}$$

where $N_Q\big(t, v(t)\big)\big(u(t)\big)$ denotes the normal cone to $Q(t, v(t))$ at the point $u(t)$, the sets $Q(t, v(t))$ are nonconvex in H and $F : [0, T] \times H \times H \to H$ is an upper semicontinuous convex valued mapping playing the role of a perturbation to the problem, that is an external force applied on the system. This kind of problems was initiated by J.J. Moreau (see [14]) for time-dependent sets $Q(t)$ and $F \equiv \{0\}$ to deal with problems arising in elastoplasticity, quasistatics, electrical circuits, hysteresis and dynamics. Since then, various generalizations have been obtained, see for instance [4-9, 16-18] and the references therein.
When the moving set Q depends also on the state, one obtain a generalization of the classical sweeping process known as the state-dependent sweeping process. Such problems are motivated by parabolic quasi-variational inequalities arising e.g. in the evolution of sandpiles, and occur also in the treatment of 2-D or 3-D quasistatic evolution problems with friction, as well as in micro-mechanical damage models for iron materials with memory to describe the evolution of the plastic strain in presence of small damages. We refer to [12] for more details. By means of a generalized version of the Shauder's fixed point theorem, Castaing, Ibrahim and Yarou [9] provided an approach to prove the existence of solution to (P). The approach is based on the Moreau's catching-up algorithm. For recent results in the study of state-dependent sweeping process, we refer to [1], [2], [11].

Our aim in this paper is twofold: using some recent concepts of set's regularity, we show how the approach from [9] can be adapted to yield the existence of solution for (P) with the general class of *equi-uniformly subsmooth* sets $Q(t, x)$. Moreover, we weaken the usual assumptions on the perturbation by taking F unnecessarily bounded and without any compactness conditions.

2. NOTATION AND PRELIMINARIES:

We denote by B the unit closed ball of the Hilbert space H, $C_H([0, T])$ the Banach space of all continuous mappings u : [0, T] → H endowed with the norm of uniform convergence. For a nonempty closed subset S of H, we denote by d(·, S) the usual distance function associated with S, $Proj_S(u)$ the projection of u onto S defined by $Proj_S(u) = \{y \in S : d(u, S) = \|u - y\|\}$. We denote by co(S) the closed convex hull of S, characterized by $co(S) = \{x \in H: \forall x' \in H, \langle x',x\rangle \leq \delta^*(x',S)\}$, where $\delta^*(x',S)= Sup_{y\in S}\langle x',y\rangle$ stands for the support function of S at x′ ∈ H. Recall that for a closed convex subset S, we have $d(x,S) = Sup_{x'\in B}[\langle x',x\rangle - \delta^*(x',S)]$. A subset S is said to be relatively ball compact, if for any closed ball B(x, r) of H,the set B(x, r) ∩ S is relatively compact.
If φ is a locally-Lipschitz function defined on *H,* the Clarke subdifferential $\partial^C\varphi(x)$ of φ at x is the nonempty convex compact subset of *H,* given by

$$\partial^C\varphi(x) = \{ \xi \in H: \varphi^\circ(x; v) \geq \langle \xi, v\rangle, \forall v \in H \},$$

where $\varphi^\circ(x; v) = lim_{y\to x}\mathrm{Sup}_{t\downarrow 0}\dfrac{\varphi(y+tv)-\varphi(y)}{t}$ is the generalized directional derivative of φ at x in the direction v (see [10]). The Clarke normal cone $N^C(S, x)$ to S at $x \in S$ is defined by polarity with T_S^C, that is, $N^C(S, x) = \{\xi \in H: \langle \xi, v\rangle \leq 0, \forall v \in T_S^C\}$, where T_S^C denotes the Clarke tangent cone and is given by $T_S^C = \{v \in H: d^\circ(x, S; v) = 0\}$.
A vector $v \in H$ is said to be in the *Fréchet* subdifferential $\partial^F\varphi(x)$ of φ at x (see [15]) provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in B(x, \delta)$
$\langle v, y - x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon\|y - x\|$.
It is known that, we have always $\partial^F\varphi(x) \subset \partial\varphi(x)$, and for all $x \in S$, $N^F(S, x) \subset N^C(S, x)$ and $\partial^F d(x, S) = N^F(S, x) \cap B$. Another important property is that, whenever $y \in Proj_S(x)$, one has $x - y \in N^F(S, y) \Rightarrow x - y \in N^C(S, y)$.

Let Ω be a closed subset of *H*, we say that Ω is subsmooth at $x \in \Omega$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle \xi_1 - \xi_2, x_1 - x_2\rangle \geq -\varepsilon \|x_1 - x_2\| \qquad (1)$$

whenever $x_1, x_2 \in B(x, \delta) \cap \Omega$ and $\xi_i \in N^C(\Omega, x_i) \cap B$, $i = 1,2$. The set Ω is subsmooth, if it is subsmooth at each point of Ω. We further say that Ω is uniformly subsmooth, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (1) holds for all $x_1, x_2 \in \Omega$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N^C(\Omega, x_i) \cap B$.
Definition 2.1 Let $(S(q))_{q\in Q}$ be a family of closed sets of H with parameter q ∈ Q. This family is called equi-uniformly subsmooth, if, for every ε > 0, there exists δ > 0 such that, for each q ∈ Q, the inequality (1) holds, for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and for all $\xi_i \in N^C(S(q), x_i) \cap B$, $i = 1,2$. For the proofs of the next proposition, we refer the reader to [3] and [19].
Proposition 2.2 Let {C(t, v) : (t, v) ∈ [0, T]×H} be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real η > 0. Assume that there exist real constants $L_1 > 0$ and $L_2 > 0$ such that, for any x, y, u, v ∈ H and s, t ∈ [0, T]

$$|d(x, C(t, u)) - d(y, C(s, v))| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Then the following assertions hold:
(a) For all (s, v; y) ∈ Gph(C), we have $\eta\,\partial d(y, C(s,v) \subset \eta B$;
(b) The convex weakly compact valued mapping $(t, x, y) \to \partial d(y, C(t,x)$ satisfies the upper semicontinuity property: For any sequence $(s_n)_n$ in [0, T] converging to s, any sequence $(v_n)_n$ converging to v, any sequence $(y_n)_n$ converging to y ∈ C(s, v) with $(y_n \in C(s_n, v_n)$, and any ξ ∈ H, we have

$$lim\ sup_{n\to\infty}\ \sigma(\xi, \eta\ \partial d(y_n, C(s_n, v_n))) \le \sigma(\xi, \eta\ \partial d(y, C(s,v))).$$

3. MAIN RESULT

Theorem 3.1 Let Q : [0, T]×H → H be a set-valued mapping with nonempty values satisfying:
(Q_1) the family {Q(t, x); (t, x) ∈ [0, T] × H} is equi-uniformly subsmooth;
(Q_2) for any bounded subset A ⊂ H, the set Q([0, T] × A) is relatively ball compact;
(Q_3) there are real constants $\Lambda_1 > 0$, and $\Lambda_2 > 0$, such that for all t, s ∈ [0, T] and x_i, y_i, $z_i \in H$

$$|d(z_1, Q(t, x_1)) - d(z_2, Q(t, x_2)| \le \|z_1 - z_2\| + \Lambda_1\ |t - s| + \Lambda_2\|x_1 - x_2\|.$$

Let F : [0, T] × H × H → H be an upper semicontinuous set-valued mapping with nonempty closed convex values such that:
(F_1) for some real κ > 0 and for all (t, x, y) ∈ [0, T] × H × H, d(0, F(t, x, y)) ≤ κ(1 + ‖x‖ + ‖y‖).
Then, for every (a, b) ∈ H × H with a ∈ Q(0, b) there exists a Lipschitz continuous solution (u, v) to (P).

Proof.

Step 1: for each *(t, x, y)* ∈[0, *T*] × *H* × *H*, denote by $m(t, x, y)$ the element of minimal norm of the closed convex set *F(t, x, y)* of *H,* that is $m(t, x, y) = Proj_{F(t,x,y)}(0)$. For every $n \ge 1$, we consider a partition of [0, *T*] by the points $t_k^n = ke_n$, $e_n = \frac{T}{n}$, $k = 0,1,2, ..., n$.

Starting from $u_0^n = a \in Q(0, b) = Q(t_0^n, v_0^n)$ and taking $u_1^n \in Proj_{Q(t_1^n,v_0^n)}(u_0^n - e_n\ m(t_0^n, u_0^n v_0^n))$ thanks to the ball compactness of the set $Q(t_1^n, v_0^n)$, let define inductively the sequences $(u_k^n)_{0\le k\le n}$ and $(v_k^n)_{0\le k\le n}$ satisfying

$$u_{k+1}^n \in Q(t_{k+1}^n, v_k^n) \quad (2)$$

$$u_{k+1}^n \in Proj_{Q(t_{k+1}^n,v_k^n)}(u_k^n - e_n\ m(t_k^n, u_k^n v_k^n)) \quad (3)$$

$$v_{k+1}^n = v_k^n + e_n\ u_{k+1}^n \quad (4)$$

$$\|u_{k+1}^n - u_k^n\| \le \Lambda_1 e_n + \Lambda_2\|v_k^n - v_{k-1}^n\| + 2e_n\ \|m(t_k^n, u_k^n, v_k^n)\|$$

$$\|v_k^n - v_{k-1}^n\| \le e_n\ \|u_k^n\|$$

and

$$\|u_k^n\| \le ((\Lambda_1 + 2\ \kappa\ (1 + \|v_0^n\|))T + \|u_0^n\|)\ e^{T(\Lambda_2 + 2\kappa(1 + 2T))} = \Delta.$$

$$\|v_k^n\| \le \|v_0^n\| + T\ \Delta = Y$$

$$\frac{\|u_{k+1}^n - u_k^n\|}{e_n} \le \Lambda_1 + \Lambda_2\Delta + 2\ \kappa\ (1 + \|v_0^n\| + (T + 1)\Delta) = \Theta. \quad (5)$$

Step 2: construction of approximate solutions $u_n(\cdot)$ and $v_n(\cdot)$. For any $t \in [t_k^n, t_{k+1}^n]$, $k \le n-1$, we define

$$v_n(t) = v_k^n + (t - t_k^n)u_{k+1}^n$$

and

$$u_n(t) = \frac{t_{k+1}^n - t}{e_n}u_k^n + \frac{t - t_k^n}{e_n}u_{k+1}^n.$$

Thus, for almost all $t \in [t_k^n t_{k+1}^n]$, $\dot{u}_n(t) = \frac{u_{k+1}^n - u_k^n}{e_n}$ and

$$-\dot{u}_n(t) \in N_{Q(t_{k+1}^n,v_k^n)}(u_{k+1}^n) + m(t_k^n, u_k^n, v_k^n)$$

Using the notations

$$p_n(t) = \begin{cases} t_k^n & if\ t \in [t_k^n, t_{k+1}^n[\\ t_{n-1}^n & if\ \ t = T \end{cases} \quad \text{and} \quad q_n(t) = \begin{cases} t_{k+1}^n & if\ t \in [t_k^n, t_{k+1}^n[\\ t_n^n & if\ \ t = T. \end{cases}$$

We can write

$$-\dot{u}_n(t) \in N_{Q(q_n(t),\ v_n(p_n(t)))}(u_n(q_n(t))) + m(p_n(t), u_n(p_n(t)), v_n(p_n(t)))$$

for a.e. *t* ∈[0, *T*]. Obviously, for all $n \ge 1$ and for all *t* ∈[0,*T*], the following hold:

$$\|m(p_n(t), u_n(p_n(t)), v_n(p_n(t)))\| \le \kappa\ (1 + \|v_0^n\| + (T+1)\ \Delta) = \Lambda$$

$$u_n(q_n(t)) \in Q(q_n(t), v_n(p_n(t)))$$

$$v_n(t) = b + \int_0^t u_n(p_n(s))ds, \quad \forall\ t \in [0, T].$$

Thanks to the ball compactness assumption and by Ascoli's Theorem, $(u_n(\cdot))$ is relatively compact in $C_H([0, T])$, so we can extract from it a subsequence, that we do not relabel, which converges uniformly to some mapping $u(\cdot) \in C_H([0, T])$. By the inequality (5) there exists a subsequence (again denote by) $(\dot{u}_n(.))$ which converges $\sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ in $L_H^1([0,T])$, *to* $\dot{u}$with $\|\dot{u}(t)\| \le \Theta$ a.e. $t \in [0,T]$.

Putting $m(p_n(\cdot), u_n(p_n(\cdot)), v_n(p_n(\cdot))) = (f_n(\cdot))$,$(f_n(\cdot))$ is bounded, taking a subsequence if necessary, we may conclude that $(f_n(\cdot))$ converges $\sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ to some mapping$f \in L_H^1([0,T])$ with $\|f(t)\| \le \Lambda$.

Step 3: the limit satisfies the inclusion.Using Mazur's theorem and Proposition 2.2, we can conclude that

$$-\dot{u}_n(t) \in N_{Q(t,v(t))}(u(t)) + f(t) \quad a.e. \quad t \in [0, T]$$
$$f(t) \in F(t, u(t), v(t)) \qquad a.e. \quad t \in [0, T].$$

□

4. APPLICATION

As an application, let consider the antiplane frictional contact problem, the friction being modeled with Tresca's law, the classical model of the process is the following:

Find a displacement field $u : \Omega \times [0, T] \to R$ such that

$$div\,(\square\nabla\dot{u} + \theta\nabla u) + f_0 = 0 \qquad \text{in} \qquad \Omega \times (0, T),$$
$$u = 0 \qquad \text{on} \qquad \Gamma_1 \times (0, T),$$
$$\square\partial_\nu\dot{u} + \theta\,\partial_\nu\,u = f_2 \text{on} \qquad \Gamma_2 \times (0, T),$$
$$\left.\begin{array}{l} |\theta\partial_\nu\dot{u} + \mu\,\partial_\nu\,u| \le g \\ \theta\partial_\nu\dot{u} + \mu\,\partial_\nu\,u = -g\,\frac{\dot{u}}{|\dot{u}|} \quad if\ \dot{u} \neq 0 \end{array}\right\} on \quad \Gamma_3 \times (0, T),$$
$$u(0) = u_0 \text{in} \qquad \Omega$$

We refer to [13] for the physical interpretation and the following variational formulation of the problem:

Find $u : I := [0, T] \to R^d$such that$\dot{u}(t) \in \Gamma$ a.e. $t \in I$ and$\forall\ v \in \Gamma$

$$a(u(t), v - \dot{u}(t)) + b\,\dot{u}(t)), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \ge \langle f(t, u(t)), v - \dot{u}(t) \rangle$$
$$u(0) = u_0 \in R^d,$$

where $a(\cdot, \cdot)$ and $b(\cdot, \cdot) : H \times H \to R$ are two real continuous bilinear and symmetric forms. See also [1] for a similar problem. Following [1], one proves the equivalence between this variational inequality and the perturbed state-dependent sweeping process.

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THE ATTRACTION BASINS OF SEVERAL ROOT FINDING METHODS, WITH A NOTE ABOUT OPTIMAL METHODS

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ABSTRACT

Finding the solution of the equation f(x)=0 when f(x) is nonlinear is very important, as like this equation resulting out from many real life problems and applied sciences. Many iterative methods were proposed to solve nonlinear equations. These methods can be compared using different ways, for example; their convergence order, number of functions needed to be evaluated in each iteration, number of iterations needed for convergence, the CPU time required to achieve the accuracy needed, and efficiency index. In this work we use another way called the basins of attraction of the method. We consider six different methods of different orders and graph the attraction basins of the roots of several polynomials. Finally, we clarify the answer to the question: are the optimal methods always good for finding the solution of the nonlinear equations?

Keywords: Basin of attraction; Nonlinear equations; Iterative methods

1. INTRODUCTION

Let $f(x)$ be nonlinear, solving the equation $f(x) = 0$ has been studied very widely, see for example [3-5,7] and the references therein. Besides, one of the most common ways to compare the efficiency of iterative methods is the efficiency index which can be determined by $q^{\frac{1}{r}}$, where q is convergence order of the iterative scheme and r represents number of functions needed to be found at each iteration. Kung and Traub[2] mentioned a conjecture says that the iterative scheme with number of functional evaluations equals r is optimal if its order of convergence equals 2^{r-1}. There are many ways to compare the efficacyof the iterative methods. The attraction basins for complex Newton's method firstly considered and attributed by Cayley[1] is a method to illustrate how different starting points affect the behavior of the function. In this way, we can compare different root finding methods by their area of convergence shown by the attraction basins of the roots. Based on that, the iterative method is better if it has larger area of convergence. Stewart [6] compared Newton method, Halley's method, Popovski method, and Leguerre method by showing the attraction basins of the zeros found by the methods. Many researchers have compared different orders iterative methods for finding multiple zeros when their multiplicity is known.

In this work we compare six different iterative methods by illustrating their attraction basins. Three of the compared iterative techniques are optimal. We try to answer the question: are the optimal schemes always good for solving nonlinear equations? The work in this study is divided as follows: we illustrate some definitions and preliminaries in Section 2. In Section 3, the basins of attractions were used to compare six different iterative methods on some polynomials. Eventually, the conclusion given in Section 4.

2. PRELIMINARIES

Firstly, let's start with some preliminaries and definitions which are related to the subject

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of basins of attraction.

If $f(x_0) = x_0$, then x_0 is called a fixed point. For $x \in \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, we define its orbit as orb$(x) = \{x, f(x), f^{[2]}(x), \ldots, f^{[n]}(x), \ldots\}$, where $f^{[n]}$ is the n^{th} iterate of f. x_0 is called a periodic point of period n if n is the smallest number such that $f^{[n]}(x_0) = x_0$. If x_0 is periodic of period n then it is a fixed point for $f^{[n]}$. A point x_0 is said to be attracting if $|f'(x_0)| < 1$, repelling if $|f'(x_0)| > 1$, and neutral if $|f'(x_0)| = 1$. Moreover, if the derivative is zero then the point is called super-attracting.

The Julia set of a nonlinear function $f(x)$, denoted by $J(f)$, is the closure of the set of its repelling periodic points. The complement of $J(f)$ is called the Fatou set $F(f)$. If O is an attracting periodic orbit of period m, we define the basin of attraction to be the open set $A \in \hat{\mathbb{C}}$ consisting of all points $x \in \hat{\mathbb{C}}$ for which the successive iterates $f^{[m]}(x), f^{[2m]}(x), \ldots$ converge towards some point of O. In symbols, we can define the basin of attraction for any root α of f to be $B(\alpha) = \{x_0 | \lim_{n\to\infty} f^{[n]}(x_0) = \alpha\}$. The basin of attraction of a periodic orbit may have infinitely many components. It can be said that basin of attraction of any fixed point tend to an attractor belonging to Fatou set, and the boundaries of these basin of attraction belongs to the Julia set. While an n order complex polynomial with distinct roots partitions the complex plane into n number of basins, the partitions may or may not be equally distributed or even connected for that matter. In an ideal setting, these attracting regions resemble a Voronoi diagram showing all points that are the nearest neighbors to the polynomial's zero. See 1[6].

3. NUMERICAL EXAMPLES

In this section we will compare various root finding methods by visualizing the basins of attraction of their zeros. All examples are about polynomials with roots of multiplicity one. We will consider six methods of different orders of convergence. Two of them were considered by Stewart [6]. The methods we consider with their order of convergence are:

1. **Newton's Method:** It is of order two, and given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

2. **Halley's Method:** It is of order three, and it is given by

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}.$$

3. **Jarratt's Method:**It is a two step method of order four, given by

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{1}{2}\dfrac{f(x_n)}{f'(x_n)} - \dfrac{1}{2}\dfrac{\frac{f(x_n)}{f'(x_n)}}{1 + \frac{3}{2}\left(\frac{f'(y_n)}{f'(x_n)} - 1\right)}. \end{cases}$$

4. **Xiaofeng-Wang Method (XW):** It is of order four method of three steps [8], for $\lambda = 0.1$ the method is given by

$$\begin{cases} z_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ y_n = z_n - \dfrac{(z_n - x_n)^2}{10}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{2f[x_n, y_n] - f'(x_n)}, \end{cases}$$

where $f[x_n, y_n] = \frac{f(x_n)-f(y_n)}{x_n-y_n}$.

5. **MBM Method:** It is an optimal three steps iterative method of order $\mathbf{8}$[3]. For $\boldsymbol{\beta} = \mathbf{0}$ the method is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n)}{f(x_n) - 2f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(x_n)} \left(\dfrac{u(-8v - 5) - v^2 + 2v - 5}{12v - 5}\right), \end{cases}$$

where $u = \frac{f(z_n)}{f(y_n)}$, and $v = \frac{f(y_n)}{f(x_n)}$. The first two steps of this method represents the well-known Ostrowski's method.

6. **Srivastava Method (SM):** It is a method of order $\mathbf{15}$[5]. The method is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right] \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f\left(z_n - \frac{f(z_n)}{f'(z_n)}\right)}{f'(z_n)}. \end{cases}$$

In the following are examples of different polynomials with different coefficients of different orders, we will plot the basins of attraction of the roots of these polynomials using the methods mentioned above.In all examples, a 4 by 4 square region is centered at the origin covering all the zeros of the tested polynomials.

A 400×400 uniform grid in the square is taken to unfold initial points for the iterative methods via basins of attraction. Each grid point of a square is colored according to the iteration number for convergence and the root it converges to. The exact roots were assigned as a black points on the graph. The appearance of darker region shows that the method requires a fewer number of iterations.All calculations have been performed on intel Core i7-3770 CPU @3.40 GHz with 4GB RAM, with Microsoft Windows 10, 64 bit based on X64-based processor. The software used to do the graphs is Mathematica 9.

Example 3.1 Consider the polynomial $\boldsymbol{f_1(x) = x^3 - 1}$ which has roots $\mathbf{1}$, $\mathbf{-0.5 \pm 0.866025}\boldsymbol{i}$.The basins of attraction for each root were illustrated in Figure1. As it can be seen, Halley's method attains larger area of convergence, followed by Jarratt's, XW and Newton's methods, while MBM and SM methods show more chaos.

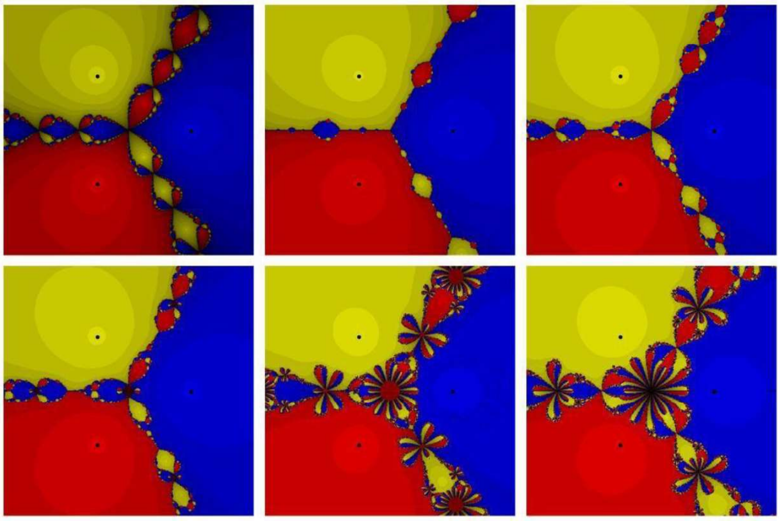

Figure 1. The basins of attraction of the roots of the polynomial$f_1(x) = x^3 - 1$.
The top row from left to right: Newton's, Halley's, Jarratt's.
The below row fromleft to right: XW, MBM and SM methods.

Example 3.2 The second example is the polynomial $\boldsymbol{f_2(x) = x^4 - \frac{5}{4}x^2 + \frac{1}{4}}$ which has four simple real roots $\boldsymbol{x = \pm 1, \pm 0.5}$.It is clear from Figure2 that Newton's, Halley's, Jarratt's and XW methods give better results than MBM method. The worst result was for SM method where a lot of black (Divergent) points appeared.

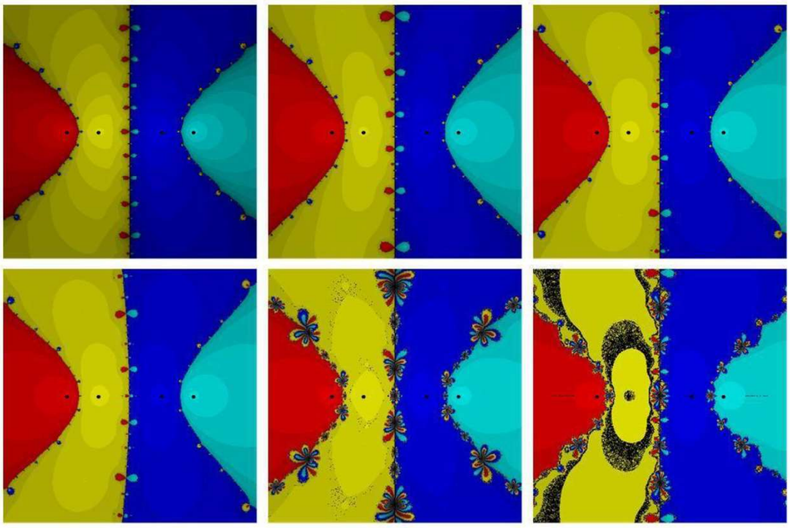

Figure 2: The basins of attraction of the roots of the polynomial$f_2(x) = x^4 - \frac{5}{4}x^2 + \frac{1}{4}$.
The top row from left to right: Newton's, Halley's, Jarratt's.
The below row fromleft to right: XW, MBM and SM methods.

Example 3.3 The four roots of unity polynomial $f_3(x) = x^4 - 1$ has the roots $x = \pm 1, \pm i$.The results from Figure3 show that Halley's method gave the best results with larger area of convergence, followed by Jarratt's, XW and Newton's methods. Again, MBM and SM methods show more divergent points.

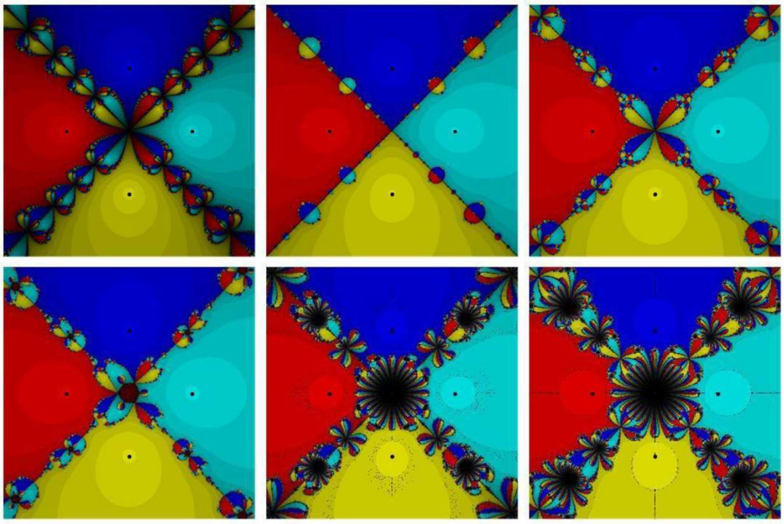

Figure 3: The basins of attraction of the roots of the polynomial$f_3(x) = x^4 - 1$.
The top row from left to right: Newton's, Halley's, Jarratt's.
The below row fromleft to right: XW, MBM and SM methods.

Table 1 presents the CPU time needed to obtain the basins of attraction of the roots of the examples considered. It is clear that there is a relation between the CPU time and the chaos in the graph, that is, less time tends to larger area of convergence and less chaos, and vice versa.

Table 1: CPU time needed in seconds.

Method	$f_1(x)$	$f_2(x)$	$f_3(x)$
Newton	6.69	7.52	7.88
Halley	5.61	6.5	6.2
Jarratt	5.63	6.53	6.44
XW	19.89	6.83	21.39
MBM	23.8	59.94	58.06
SM	31.98	34.14	48.45

3.1. How good are optimal methods for nonlinear equations?

According to the conjecture of Kung and Traub[2], from the six compared iterative methods mentioned above, we have three optimal methods, Jarratt, XW, and MBM methods. Its clear from the basins of attraction of these methods that if the iterative method is optimal then it is not essential that it has better attraction basins (larger area of convergence), see MBM attraction basins in all examples. Also, if two optimal methods are of the same order, then its not necessary that they have the same basins of attraction. Both Jarratt's method and XW method are optimal of order four, but Jarratt's method is look like that it has larger area of convergence than XW method. Note that although Jarratt's method and XW method have very close basins of attraction in most examples above, but the CPU time needed to draw their basins of attraction is widely different, see Table 1 for the functions f_1 and f_3.

Based on what we mentioned above we can answer the following question: Are the optimal methods always better for solving nonlinear equations than other methods? The answer is clearly No. Even the optimal methods need less functional evaluations in each iteration, but its clear from the basins of attraction in the previous examples that sometimes optimal methods show a lot of chaos, which means number of divergent points is greater some times in optimal methods than other non-optimal methods. One can conclude that number of functional evaluations in each iteration is not the only factor that confirm the efficiency of the iterative technique, there are other factors that affect also, like number of steps in the iterative scheme, order of convergence, and number of arithmetic operations needed at each iteration.

4. CONCLUSION

In this paper we have considered six different schemes of different orders for solving nonlinear equations. It can be concluded that obtaining better basins of attraction is not depending on the order of convergence of the method.Also, one can note that the optimality property of iterative method is not always good for solving nonlinear equations, as the area of convergence of the roots of the function not depends only on number of functional evaluations in each iteration, but there are many other factors like number of steps in the iterative scheme, order of convergence, and number of arithmetic operations needed at each iteration.

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(S,T)-NORMED DOUBT NEUTROSOPHIC IDEALS OF *BCK/BCI*-ALGEBRAS

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ABSTRACT

In this paper, the ideaof (S,T)- normed doubt neutrosophic ideals of BCK/BCI-algebras is introduced and the characteristic properties are described. Then, images and preimages of (S,T)- normed doubt neutrosophic ideals under homomorphism are considered.

Keywords: BCK/BCI-algebra; doubt neutrosophic ideal; (S,T)-normed doubt neutrosophic ideal

1. INTRODUCTION

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [4], and they have been applied to several domains such as groups, rings, topology and measure theory. Additionally, Iséki [5] initiated the idea of aBCI-algebra, which is a generalization of a BCK-algebra. The idea of neutrosophic set theory proposed by Smarandache[11, 12] is a more general platform that extends the ideas of ordinary, fuzzy and intuitionistic fuzzy sets, and that is used to several parts: decision making, pattern recognition and medical diagnosis. Triangular norms were proposed by Schweizer and Sklar[10] to model the distances in probabilistic metric spaces. In fuzzy sets, t-conorm (S) and t-norm (T) are extensively applied to model the logical connectives: conjunction (AND) and disjunction (OR). There are several applications of triangular norms in many domains of artificial intelligence [5] and mathematics. The first definition of fuzzy subalgebras and ideals in BCK/BCI-algebras was by Xi [13] in 1991. Modifying Xi's definition, Jun [6] in 1994 presented doubt fuzzy subalgebras and ideals in BCK/BCI-algebras. After that, many other researchers used this idea and published several articles in different branches of algebraic structures [1, 7,14].Motivated by the previous studies, we present the notion of (S,T)- normed doubt neutrosophicideals of BCK/BCI-algebras and describe some of the characteristic properties.Then, we consider images and preimages of (S,T)- normed doubt neutrosophic ideals under homomorphism.

2. PRELIMINARIES

During this paper, let X be a BCK/BCI-algebra unless otherwise specified.

A structure $(X,*)$ is called a BCK-algebra (see [4]) if X contains a special element 0 andsatisfies the following axioms for all $\mathrm{x},\mathrm{y},\mathrm{z} \in X$:

I. $((x*y)*(x*z))*(z*y)=0$,
II. $(x*(x*y))*y=0$,
III. $x*x=0$,
IV. $x*y=0$ and $y*x=0$ imply $x=y$.

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If a BCI-algebra X satisfies $0*x=0$, then X is called a BCK-algebra. In a BCK/BCI-algebra, $x*0=x$ holds. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y=0$. A non-empty subset J of X is called an ideal of X if for all $x,y \in X$,(1) $0 \in J$, (2) $x*y \in J$ and $y \in J$ imply $x \in J$.

Definition 2.1. [11,12]A neutrosophic set in a non-empty set $X$is a structure of the form:

$$B = \{\langle x; B_T(x), B_I(x), B_F(x)\rangle | x \in X\},$$

where $B_T, B_I, B_F: X \to [0,1]$. We shall use the symbol $B = (B_T, B_I, B_F)$ for the neutrosophic set $B = \{\langle x; B_T(x), B_I(x), B_F(x)\rangle | x \in X\}$.

Definition 2.2.[10]A function$T: [0,1] \times [0,1] \to [0,1]$is called a triangular norm, if it satisfiesthe following conditions: for all$x, y. z \in [0,1]$,
(1) $T(0,0) = 0, T(1,1) = 1$,
(2) $T(x, T(y, z)) = T(T(x, y), z)$,
(3) $T(x, y) = T(y, x)$,
(4) $T(x, y) \leq T(x, z)$ if $y \leq z$.

If $T(x, 0) = x$ and $T(x, 1) = x$ for all $x \in [0,1]$, then T is called a t-conorm and a t-norm, respectively. Throughout this paper, denote S and T as a t-conorm and a t-norm, respectively. Some examples of t-conorms and t-norms are
- $S_M(x, y) = \max\{x, y\}$ and $T_M(x, y) = \min\{x, y\}$.
- $S_L(x, y) = \min\{x + y, 1\}$ and $T_L(x, y) = \max\{x + y - 1, 0\}$.
- $S_P(x, y) = x + y - xy$ and $T_P(x, y) = xy$.

A t-conorm S and a t-norm T are called associated [11], i.e., $S(x, y) = 1 - T(1 - x, 1 - y)$, for all $x, y \in [0,1]$.

Lemma 2.3. [3] *For any* $x, y \in [0,1]$, *we have*$0 \leq \max\{x, y\} \leq S(x, y) \leq 1$*and* $0 \leq T(x, y) \leq \min\{x, y\} \leq 1$.

Definition 2.4.[14]A fuzzy set μ of X is called a doubt fuzzy ideal of$X$if$\mu(0) \leq \mu(x) \leq \max\{\mu(x * y), \mu(y)\}$forall$x, y \in X$.

3. $(\mathbf{S}, \mathbf{T})$-NORMED DOUBT NEUTROSOPHIC IDEALS

Definition 3.1.A neutrosophic set$B = (B_T, B_I, B_F)$of$X$is called a doubt neutrosophic idealof$X$if forall$x, y \in X$,
(1) $B_T(0) \leq B_T(x) \leq \max\{B_T(x * y), B_T(y)\}$,
(2) $B_I(0) \leq B_I(x) \leq \max\{B_I(x * y), B_I(y)\}$,
(3) $B_F(0) \geq B_F(x) \geq \min\{B_F(x * y), B_F(y)\}$.

Definition 3.2.A neutrosophic set$B = (B_T, B_I, B_F)$of$X$is called a doubt neutrosophic idealof$X$with respect to at-conorm$S$anda$t$-norm$T$(or simply,an(S, T)-normed doubt neutrosophic ideal of$X$)if for all$x, y \in X$,
(1) $B_T(0) \leq B_T(x) \leq S(B_T(x * y), B_T(y))$,
(2) $B_I(0) \leq B_I(x) \leq S(B_I(x * y), B_I(y))$,
(3) $B_F(0) \geq B_F(x) \geq T(B_F(x * y), B_F(y))$.
Example 3.3. Consider aBCK-algebra$X = \{0, k, l, m\}$which is defined in Table 1:

Table 1: The operation $*$

$*$	0	k	l	m
0	0	0	0	0
k	k	0	0	k
l	l	k	0	l
m	m	m	m	0

Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 2:

Table 2: Neutrosophic set $B = (B_T, B_I, B_F)$

X	$B_T(x)$	$B_I(x)$	$B_F(x)$
0	0	0	1
k	0.50	0.40	0.33
l	0.50	0.40	0.33
m	1	0.90	0

Clearly, $B_T(0) \leq B_T(x) \leq S_M(B_T(x * y), B_T(y)), B_I(0) \leq B_I(x) \leq S_M(B_I(x * y), B_I(y))$ and $B_F(0) \geq B_F(x) \geq T_L(B_F(x * y), B_F(y))$ for all $x, y \in X$. Hence, $B = (B_T, B_I, B_F)$ is an (S_M, T_L)-normed doubt neutrosophic ideal of X. Also, a t-conorm S_M and a t-norm T_L are not associated.

Remark 3.4. *Example 3.3 holds even with the t-conorm S_M and t-norm T_M. Hence, $B = (B_T, B_I, B_F)$ is an (S_M, T_M)-normed doubt neutrosophic ideal of X.*

Remark 3.5. *Every doubt neutrosophic ideal of X is an (S, T)-normed doubt neutrosophic ideal of X, but the converse is not true.*

Example 3.6. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ which is defined in Table 3:

Table 3: The operation $*$

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 4:

Table 4: Neutrosophic set $B = (B_T, B_I, B_F)$

X	$B_T(x)$	$B_I(x)$	$B_F(x)$
0	0.50	0.50	0.33
1	0.50	0.50	0.33
2	0.50	0.50	0.33
3	0.75	0.75	0.25
4	0.75	0.75	0.25

Clearly, $B_T(0) \leq B_T(x) \leq S_L(B_T(x * y), B_T(y)), B_I(0) \leq B_I(x) \leq S_L(B_I(x * y), B_I(y))$ and $B_F(0) \geq B_F(x) \geq T_P(B_F(x * y), B_F(y))$ for all $x, y \in X$. Hence, $B = (B_T, B_I, B_F)$ is an (S_L, T_P)-normed doubt neutrosophic ideal of X, but it is not a doubt neutrosophic ideal of X.

Definition 3.7. A mapping $\theta: X \to Y$ of BCK/BCI-algebras is said to be a homomorphism if $\theta(x * y) = \theta(x) * \theta(y)\ \forall x, y \in X$. If $\theta: X \to Y$ is a homomorphism, then $\theta(0) = 0$.
Let $\theta: X \to Y$ be a homomorphism of BCK/BCI-algebras. For any neutrosophic set $B = (B_T, B_I, B_F)$ in Y, we define a new neutrosophic set $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ such that for all $x \in X$,

$$B_T[\theta]: X \to [0,1], B_T[\theta](x) = B_T(\theta(x)),$$
$$B_I[\theta]: X \to [0,1], B_I[\theta](x) = B_I(\theta(x)),$$
$$B_F[\theta]: X \to [0,1], B_F[\theta](x) = B_F(\theta(x)).$$

Theorem 3.8. *Let $\theta: X \to Y$ be a homomorphism of BCK/BCI-algebras. If $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y, then $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X.*

Proof. We first have

$$B_T[\theta](0) = B_T(\theta(0)) = B_T(0) \leq B_T(\theta(x)) = B_T[\theta](x),$$
$$B_I[\theta](0) = B_I(\theta(0)) = B_I(0) \leq B_I(\theta(x)) = B_I[\theta](x),$$
$$B_F[\theta](0) = B_F(\theta(0)) = B_F(0) \geq B_F(\theta(x)) = B_F[\theta](x)$$

for all $x, y \in X$. Let $x, y \in X$. Then,

$$\begin{aligned} B_T[\theta](x) = B_T(\theta(x)) &\leq S(B_T(\theta(x) * \theta(y)), B_T(\theta(y))) \\ &= S(B_T(\theta(x * y)), B_T(\theta(y))) \\ &= S(B_T[\theta](x * y), B_T[\theta](y)), \end{aligned}$$

$$\begin{aligned} B_I[\theta](x) = B_I(\theta(x)) &\leq S(B_I(\theta(x) * \theta(y)), B_I(\theta(y))) \\ &= S(B_I(\theta(x * y)), B_I(\theta(y))) \\ &= S(B_I[\theta](x * y), B_I[\theta](y)) \end{aligned}$$

and

$$\begin{aligned} B_F[\theta](x) = B_F(\theta(x)) &\geq T(B_F(\theta(x) * \theta(y)), B_F(\theta(y))) \\ &= T(B_F(\theta(x * y)), B_F(\theta(y))) \\ &= T(B_F[\theta](x * y), B_F[\theta](y)). \end{aligned}$$

Therefore, $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X.□

Theorem 3.9. *Let $\theta: X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $B = (B_T, B_I, B_F)$ be a neutrosophic set of Y. If $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X, then $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y.*

Proof. For any $b \in Y$, there exists $a \in X$ such that $\theta(a) = b$. Then,

$$B_T(0) = B_T(\theta(0)) = B_T[\theta](0) \leq B_T[\theta](a) = B_T(\theta(a)) = B_T(b),$$
$$B_I(0) = B_I(\theta(0)) = B_I[\theta](0) \leq B_I[\theta](a) = B_I(\theta(a)) = B_I(b),$$
$$B_F(0) = B_F(\theta(0)) = B_F[\theta](0) \geq B_F[\theta](a) = B_F(\theta(a)) = B_F(b).$$

Let $x, y \in Y$. Then, $\theta(a) = x$ and $\theta(b) = y$ for some $a, b \in X$. It follows that

$$\begin{aligned} B_T(x) &= B_T(\theta(a)) = B_T[\theta](a) \\ &\leq S(B_T[\theta](a * b), B_T[\theta](b)) \\ &= S(B_T(\theta(a * b)), B_T(\theta(b))) \\ &= S(B_T(\theta(a) * \theta(b)), B_T(\theta(b))) \\ &= S(B_T(x * y), B_T(y)), \end{aligned}$$

$$\begin{aligned} B_I(x) &= B_I(\theta(a)) = B_I[\theta](a) \\ &\leq S(B_I[\theta](a * b), B_I[\theta](b)) \\ &= S(B_I(\theta(a * b)), B_I(\theta(b))) \\ &= S(B_I(\theta(a) * \theta(b)), B_I(\theta(b))) \\ &= S(B_I(x * y), B_I(y)) \end{aligned}$$

and

$$\begin{aligned} B_F(x) &= B_F(\theta(a)) = B_F[\theta](a) \\ &\geq T(B_F[\theta](a * b), B_F[\theta](b)) \\ &= T(B_F(\theta(a * b)), B_F(\theta(b))) \\ &= T(B_F(\theta(a) * \theta(b)), B_F(\theta(b))) \\ &= T(B_F(x * y), B_F(y)), \end{aligned}$$

Therefore, $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y.□

5. CONCLUSIONS

We have presented the notion of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras and described the characteristic properties. Then, we have considered images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism.

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Solutions of Burgers-Lokshin Equation with its Properties

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ABSTRACT

In this paper we shall solve Burger-Lokshin (BL) equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^\alpha u}{\partial t^\alpha} + b\frac{\partial(\frac{u^2}{2})}{\partial x} = 0$$

Where t>0

$$u(x,t)\big|_{t=0} = u^0(x)$$

c>0, > 0 ,$\alpha \in (0,1)$, b≥0

byapproximate method namely Sumudu transform. Also the statistical properties of the solution will be studied.

*Keywords:*Burger-Lokshin (BL) equation ,Fractional calculus, Caputo derivative,Sumudu transforms.

1. INTRODUCTION

The fractional differential equations(FDEs) appear more and more frequently in different research areas and engineering applications[7]. Momani[9] has presented nonperturbative analytical solutions of the space-and time-fractional Burgers equations by Adomain decomposition method. Wang[8] extend the application of the homotopy perturbation and Adomian decomposition methods to construct approximate solutions for the nonlinear fractional KdV-Burgers equation.

The one-way Burgers-Lokshin (BL) equation is the simplest model, that combines both these features , it has the following form:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^\alpha u}{\partial t^\alpha} + b\frac{\partial(\frac{u^2}{2})}{\partial x} = 0$$

Where t>0

$$u(x,t)\big|_{t=0} = u^0(x)$$

with compactly supported initial datum $u(t=0,x) = u^0(x)$ at t=0. The coefficients are c>0 the sound speed, and ε>0 which takes into account the specific length of both viscous and thermal effects and the radius of the duct, also the fractional order $\alpha \in (0,1)$ is $\alpha = \frac{1}{2}$, b≥0, or Burgers coefficient, quantifies the nonlinear effects[6].

The Sumudu transform method (STM) was first proposed by Watugala[4]. [5] the author started from the definition of the Sumudu transform on general time scales to define the discrete Sumudu transform and present its basic properties.

2. PRELIMINARIES

In this section, we present some basic definitions and properties of the fractional calculus theory and Sumudu transform which are used in this work.

Definition 2.1[3]

A real function $f(x), x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number p(p >μ) , such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, where $m \in \mathbb{N}$.

Definition 2.2[3]
The Caputo definition of fractional derivative operator is given by;

$$D_a^{\alpha} f(x) = J_a^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ t > 0 \qquad (1)$$

For $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$.

Definition 2.3 [1]
The Sumudu transform is defined as follows,
Let

$$| \ \exists M, \tau_1, \tau_2 > 0, \ |f(t)| < e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)\} \qquad (2) A = \{f(t)$$

is defined as,

$$) \qquad (3) \qquad \mathbb{S}[f(t)] = G(u) = \int_0^{\infty} f(ut) e^{-t} dt = \int_0^{\infty} \frac{1}{u} f(t) e^{\frac{-t}{u}} dt, u \in (-\tau_1, \tau_2$$

Properties of the Sumudu transform are given as:
1. $\mathbb{S}[1] = 1$;
2. $\mathbb{S}[\frac{t^n}{\Gamma(n+1)}] = u^n$, $n > 0$;
3. $\mathbb{S}[e^{at}] = \frac{1}{1-au}$;
4. $\mathbb{S}[\alpha f(x) + \beta g(x)] = \alpha \mathbb{S}[f(x)] + \beta \mathbb{S}[g(x)]$.

Theorem 2.1[1]
If $G^n(u)$ is the Sumudu transform of n-th order derivative of $f^n(t)$,for $n \geq 1$ then we have :

$$G^n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}$$

Where $-1 < n - 1 \leq \alpha < n$.

Lemma 2.1[1]
The Sumudu transform $\mathbb{S}[f(x)]$ of the fractional derivative introduced by Caputo is given by

$$\mathbb{S}[D^{\alpha} f(x)] = \frac{\mathbb{S}[f(x)]}{u^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{\alpha-k}}, \ n-1 < \alpha \leq n \qquad (4)$$

3. ANALYSIS OF THE METHOD[1]

In this work, we apply Sumudu transform method to solve nonlinear Burgers-Lokshin (BL) equation. Consider a nonlinear differential equationswith initial condition of the form:

$$D^{\alpha} y = R(y) + N(x - \tau) + g(x), \tau \in \mathbb{R}, x < \tau, n - 1 < \alpha \leq n \qquad (5)$$

$$\frac{\partial^{(\alpha)} u(x,0)}{\partial t^{\alpha}} = u^{(\alpha)}(x,0)|_{t=0} = f(x), \ \alpha = 0, 1, 2, \ldots, n-1. \quad , \qquad (6)$$

Where $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivatives, $g(x,t)$ is the source term, L is the linear operator and N is the general nonlinear operator.usingSumudu transform on both sides of equation (5)

$$\mathbb{S}[D_t^{\alpha} u(x,t)] = \mathbb{S}[\boldsymbol{L}(x,t) + \boldsymbol{N}u(x,t) + g(x,t)] \qquad (7)$$

Using the property of Sumudu transform (4) and substituting into (6) we have:

$$u^{-\alpha} \mathbb{S}[u(x,t)] - \sum_{k=0}^{m-1} u^{-(\alpha-k)} u^{(k)}(x,0) = \mathbb{S}[\boldsymbol{L}u(x,t) + Nu(x,t) + g(x,t)] \qquad (8)$$

Then,

$$\mathbb{S}[u(x,t)] = \sum_{k=0}^{m-1} u^k f_k(x) + u^{\alpha} \mathbb{S}[\boldsymbol{L}u(x,t) + \boldsymbol{N}u(x,t) + g(x,t)] \qquad (9)$$

So, the standard Sumudu decomposition method is an infinite series given by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (10)$$

The nonlinear term operator [2] is decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u) \qquad (11)$$

Where A_n are the Adomian polynomials of $u_0, u_1, \dots, u_n, \dots$ that are obtain by:
$A_n(u) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[\boldsymbol{N}(\sum_{i=0}^{\infty}\lambda^i u_i)]\,|_{\lambda=0}$, $n = 0, 1, 2,\dots$ (12)

The Adomian's polynomials for equation (12) are obtained from the following:

$$A_n = \sum_{\substack{0<i<n \\ 0<j<n}} u_i \frac{\partial u_j}{\partial x}$$

Then, substituting equations (10) and (11) into (9) to get:
$\mathbb{S}[\sum_{n=0}^{\infty} u_n(x,t)] = \sum_{k=0}^{m-1} u^k f_k(x) + u^{\alpha}\mathbb{S}[\boldsymbol{L}\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n(u) + g(x,t)]$ (13)

Comparing both sides of (13) yields the following iterative algorithm:

$$\mathbb{S}[u_0(x,t)] = \sum_{k=0}^{m-1} u^k f_k(x)$$

$\mathbb{S}[u_1(x,t)] = u^{\alpha}\mathbb{S}[\boldsymbol{L}u_0(x,t) + A_0(u(x,t)) + g(x,t)]$
$\mathbb{S}[u_{n+1}(x,t)] = u^{\alpha}\mathbb{S}\left[\boldsymbol{L}u_n(x,t) + A_n\left(u(x,t)\right)\right], \quad n \geq 1.$
Applying inverse Sumudu transform to both sides of the above equations yields:
$u_0(x,t) = \mathbb{S}^{-1}(\sum_{k=0}^{m-1} u^k f_k(x))$
$u_1(x,t) = \mathbb{S}^{-1}(u^{\alpha}\mathbb{S}\left[\boldsymbol{L}u_0(x,t) + A_0\left(u(x,t)\right) + g(x,t)\right])$
$u_{n+1}(x,t) = \mathbb{S}^{-1}(u^{\alpha}\mathbb{S}\left[\boldsymbol{L}u_n(x,t) + A_n\left(u(x,t)\right)\right]), n \geq 1.$
Finally, the solution $u_n(x, t; \alpha)$ can be approximated by the truncated series;
$u_n(x,t) = \sum_{j=0}^{n-1} u_j(x,t)$ (14)
Such that
$\lim_{n\to\infty} u_n(x,t) = u(x,t)$ (15)

Now applying Sumudu transform method to solve Burger-Lokshin (BL) equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + b\frac{\partial(\frac{u^2}{2})}{\partial x} = 0 \quad (16)$$

Where t>0, c>0, > 0 ,$\alpha \in (0,1)$, b≥0
$u(x,0) = \sin x$ (17)
Taking Sumudu transform of equation (16), and using the property of Sumudu transform together with the initial condition, we get:
$\mathbb{S}[u(x,t)] = \frac{1}{\varepsilon}u^{\alpha}\mathbb{S}[-\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} - b\frac{\partial(\frac{u^2}{2})}{\partial x})].$ (18)
The inverse of Sumudu transform implies that;
$\sum_{n=0}^{\infty} u_n(x,t) = \frac{1}{\varepsilon}\mathbb{S}^{-1}[u^{\alpha}\mathbb{S}[\sum_{n=0}^{\infty} u_n(x,t) - \sum_{n=0}^{\infty} u_n(x,t) - b(\frac{\sum_{n=0}^{\infty} u_n(x,t)}{2})^2]]$(19)
The recursive relation is given as:
$u_0(x,t) = \sin x$
$u_1(x,t) = \frac{1}{\varepsilon}\mathbb{S}^{-1}[u^{\alpha}\mathbb{S}[-\frac{\partial}{\partial t}u_0(x,t) - c\frac{\partial}{\partial x}u_0(x,t) - b\frac{\partial(\frac{u^2{}_0(x,t)}{2})}{\partial x}]]$
$\vdots$
$u_n(x,t) = \frac{1}{\varepsilon}\mathbb{S}^{-1}[u^{\alpha}\mathbb{S}[[-\frac{\partial}{\partial t}u_{n-1}(x,t) - c\frac{\partial}{\partial x}n - 1(x,t) - b\frac{\partial(\frac{u^2{}_{n-1}(x,t)}{2})}{\partial x}]\,]$

Upon passing calculations, we get:

$$u_1(x,t) = \frac{1}{\varepsilon}(\frac{-ct^{\alpha}}{\Gamma(\alpha+1)}\cos x - b\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin x\cos x)$$

$$u_2(x,t) = \frac{1}{\varepsilon}(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\cos x + \frac{b}{\varepsilon}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\sin x\cos x - \frac{c^2}{\varepsilon}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x$$

$$+\frac{bc}{\varepsilon}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}(-\sin^2 x+cos^2x)-\frac{b^2c}{\varepsilon^2}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}(-2\sin x\cos x+cos^3x)$$
$$+\frac{b^3}{\varepsilon^2}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}(\sin^3 xcos\ x)+\frac{b^3}{\varepsilon^2}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}(\sin xcos^3\ x))$$

And so on.

The solution by Sumudu transformation is:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)=u_0(x\,,t)+u_1(x\,,t)+u_2(x\,,t)+\cdots$$
$$=\sin x-\frac{c}{\varepsilon}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\cos x-\frac{b}{\varepsilon}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin x\cos x+\frac{c}{\varepsilon^2}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\cos x$$
$$+\frac{b}{\varepsilon^2}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\sin x\cos x-\frac{c^2}{\varepsilon}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x+\frac{bc}{\varepsilon^2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}(-\sin^2 x+cos^2x)$$
$$-\frac{b^2c}{\varepsilon^3}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}(-2\sin x\cos x+cos^3x)\quad(20)$$
$$+\frac{b^3}{\varepsilon^3}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}(\sin^3 xcos\ x)+\frac{b^3}{\varepsilon^3}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}(\sin xcos^3\ x)+\ldots$$

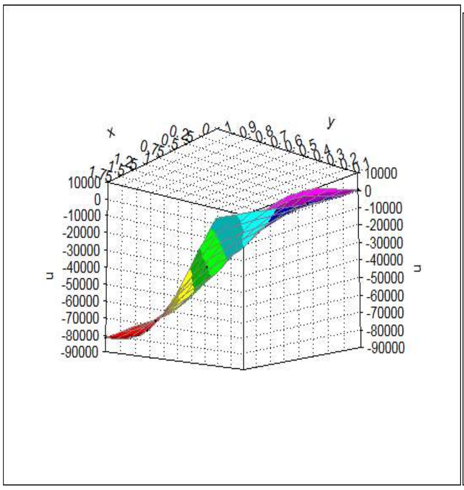

Figure 1 : represent the solution where$0<t<1,0<x<2,c=300,b=0,\varepsilon=1$and $\alpha=1/2$

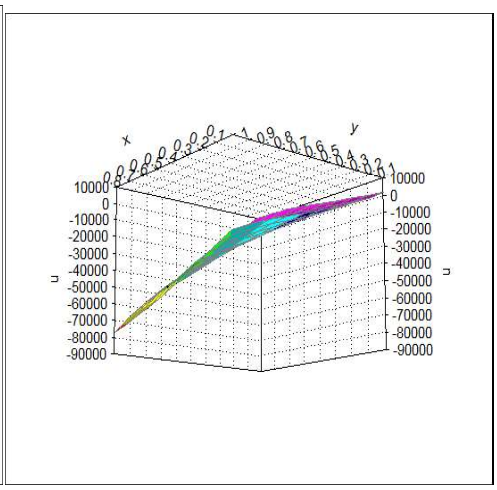

Figure 2 : represent the solution where $0<t<1,0.1<x<0.9,c=300,b=0,\varepsilon=1$and $\alpha=1/2$

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On The Solutions of Quartic Diophantine Equation with Three Variables

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ABSTRACT

Diophantine equations have a central and a significant role in mathematics especially in number theory. It is an algebraic equation or a system of polynomial equations with several variables and high order to be solved in set of integers, set of rational numbers, or other number rings. It is not easy to solve Diophantine equations if the number of variables is more than the number of equations.

The paper proposes a method to find infinitely non zero solutions of quartic diophantine equation with three unknowns in set of integers. Then, several properties for solutions are demonstrated. Also, significant relations between special numbers and solutions are determined and one of open problems in the literature is completed/solved.

Keywords: Quartic Diophantine Equation;Integer solutions of Pell Equations; Linear Transformations; Special Sequences.
2010 Mathematics Subject Classification: 11D25, 11B83, 11D09.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we consider a ternary quartic Diophantine equation given by $10x_3^4 - 11x_1^2 + 11x_2^2 = 3(x_2 + x_1)$. The main aim of the paper is to determine some non-zero integer solutions of the such non-homogenous Diophantine equation. For all non-zero integer solutions of the equation, we have to consider and apply four different patterns include different transformations. But, we just prove one pattern with a linear transformation in this work. To get integer solutions of the such Diophantine equation, we use following steps: First, we create a transformation to reduce to Pell equation and secondly , we apply Brahmagupta's Lemma on the obtained Pell Equation in the first step to have integer solutions. Also, we get various properties for the solutions in the terms of some special numbers such as Nasty numbers, Bi-quadratic numbers, Polygonal numbers, Pyramidal numbers etc...

We have used all references [1-19] to obtain our results in the paper. Especially, we require following basic notions and theories to get and prove Main Results section.

Definition 1.1.A biquadratic number is a fourth power of an integer, it means that δ^4. The first few biquadratic numbers are 1, 16, 81, 256, 625, ... It is related with Waring's problem which is defined as "Every positive integer is expressible as a sum of (at most) biquadratic integers".

Definition 1.2. (Nasty Numbers) A nasty number is a positive integer with at least four different factors such that the difference between the numbers in one pair of factors is equal to the sum of the numbers of another pair and the product of each pair is equal to the number. Thus a positive integer n is a nasty number, if n = a ∗ b = c ∗ d and a + b = c − d where a, b, c, d are positive integers.

Example 1.3. The positive integer u with four different factors is 96 and it is nasty number. Since factors of 96 = 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96 and 96 = 8 × 12 = 24 × 4 as well as 8 + 12 = 24 – 4. Therefore 96 is a nasty number.

Lemma 1.4. Properties of Nasty Numbers are given by following expressions:

1. If u is a nasty number, then clearly v^{2u} is also a nasty number for every non-zero integral value of v.
2. If four positive integers $\alpha, \beta, \gamma, \delta$ such that α, β, γ are in arithmetic progression with δ as their common difference, then $u = \alpha * \beta * \gamma * \delta$ is a nasty number.
3. Every integer u of the form $6.(1^2 + 2^2 + 3^2 + \cdots + k^2)$ is a nasty number.
4. Every integer u of the form $6.[1^2 + 3^2 + \cdots + (2k - 1)^2]$ is a nasty number.

Definition 1.5.(Polygonal Numbers) Polygonal numbers are number representing dots that are arranged into a geometric figure. Starting from a common point and augmenting outwards, the number of dots utilized increases in successive polygons. As the size of the figure increases, the number of dots used to construct it grows in a common pattern.

The concept of polygonal numbers was first defined by the Greek mathematician Hypsicles in the year 170 BC. There are some different types of polygonalnumbers such as square numbers,triangular numbers, pentagonal numbers so on..

In this work, we use Polygonal number of rank n with size m defined as follows:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right] \quad (1)$$

Definition 1.6.(Pyramidal Numbers) A figurate number corresponding to a configuration of points which form a pyramid with m-sided regular polygon bases can be thought of as a generalized pyramidal number.

In the numbers, m=3 corresponds to a tetrahedral number, and m=4 to a square pyramidal number. Pyramidal numbers may also be generalized to higher dimensions as hyperpyramidal numbers.

In this paper, we consider Pyramidal number of rank n with size m which is defined as following equation:

$$P_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)] \quad (2)$$

Lemma 1.7.(Brahmagupta's Lemma) If (x_1, y_1) is a solution of $Dx^2 + s_1 = y^2$ and (x_2, y_2) is a solution of $Dx^2 + s_2 = y^2$, then $(x_1y_2 + x_2y_1, y_2y_1 + Dx_1x_2)$ and $(x_1y_2 - x_2y_1, y_2y_1 - Dx_1x_2)$ are solutions of $Dx^2 + s_1s_2 = y^2$.

Note: In the 17th century , Fermat started to work on Pell equation in europe and after him, Euler and Lagrange continued. John Pell, after whom the Pell equation is named.

Definition 1.8.$x^2 - Dy^2 = 1$ is known as Positive Classic Pell's equation, where D i s a positive integer which is not a perfect square.

Definition 1.9.A transformation is a function from one vector space to another that respects the underlying structure of each vector space. A transformation is also known as a operator or map.Especially, linear transformations are useful since they preserve the structure of a vector space.

2. MAIN RESULTS

Theorem 2.1. Let

$$10x_3^4 - 11x_1^2 + 11x_2^2 = 3(x_2 + x_1) \qquad (3)$$

beternary quartic Diophantine equation. Then, followings are satisfied.

(i) There is a transformation that (3) equation reduces to positive Pell equation and the least positive solution of the (3) is determined by $(x_1, x_2, x_3) = (126,124,5)$.

(ii) The corresponding other non-zero integer solutions to (3) are stated by

$$x_{1_{m+1}} = \frac{63}{2}r_m^2 + \frac{2445}{88}s_m^2 + \frac{555\sqrt{22}}{44}r_m s_m \quad \text{and} \quad x_{2_{m+1}} = 31r_m^2 + \frac{2395}{88}s_m^2 + \frac{545\sqrt{22}}{44}r_m s_m$$

$$x_{3_{m+1}} = \frac{5}{2}r_m + \frac{\sqrt{22}}{2}s_m$$

such that r_m, $s_m$are defined by the solutions of positive classical Pell equation.

Proof. As we said in the introduction part there are different patterns of the solutions for (3) and in here we just use one of them as following:

(i) Let us consider the transformations

$$x_1 = 5\alpha^2 + \beta^2 \ , x_2 = 5\alpha^2 - \beta^2 \ , x_3 = \alpha \qquad (4)$$

By substituting (4) into the (3) we get following positive Pell equation.

$$\alpha^2 - 22\beta^2 = 3 \qquad (5)$$

Using a computer program (or Continued Fraction Algorithm) for finding the least positive solution of (5), we obtain

$$\alpha_0 = 5 \ \text{and} \ \beta_0 = 1 \qquad (6)$$

If (6) substitutes in (4), then following values are got for x_1, x_2, x_3.

$$x_1 = 126, \ x_2 = 124 \text{ and } x_3 = 5 \qquad (7)$$

So, the least positive solution of the (3) is attained by $(x_1, x_2, x_3) = (126,124,5)$.

(ii) For other general solutions(α_m^*, β_m^*) of positive Pell equation (5), considering the positive Pell equation$\alpha^2 - 22\beta^2 = 1$, we get general solutions as follows:

$$\alpha_m^* = \frac{1}{2}\left[\left(197 + 42\sqrt{22}\right)^{m+1} + \left(197 - 42\sqrt{22}\right)^{m+1}\right] = \frac{1}{2}r_m$$

$$\beta_m^* = \frac{1}{2\sqrt{22}}\left[\left(197 + 42\sqrt{22}\right)^{m+1} - \left(197 - 42\sqrt{22}\right)^{m+1}\right] = \frac{1}{2\sqrt{22}}s_m$$

for $m = -1, 0, 1, 2, ...$

From Lemma1.2, applying Brahmagupta's lemma between the solutions (α_0, β_0) and(α_m^*, β_m^*),the sequence of integer solutions to (5) are defined by

$$\alpha_{m+1} = \tfrac{1}{2}\left(5r_m + \sqrt{22}s_m\right) \quad \text{and} \quad \beta_{m+1} = \tfrac{1}{2}\left(r_m + \tfrac{5}{\sqrt{22}}s_m\right) \tag{8}$$

for $m = -1, 0, 1, 2, \ldots$
If we substitute (8) to (4), then general corresponding non-zero integer solutions to (3) are determined by

$$x_{1_{m+1}} = \tfrac{63}{2}r_m^2 + \tfrac{2445}{88}s_m^2 + \tfrac{555\sqrt{22}}{44}r_m s_m \quad \text{and} \quad x_{2_{m+1}} = 31r_m^2 + \tfrac{2395}{88}s_m^2 + \tfrac{545\sqrt{22}}{44}r_m s_m$$

$$x_{3_{m+1}} = \frac{5}{2}r_m + \frac{\sqrt{22}}{2}s_m$$

for $m = -1, 0, 1, 2, \ldots$

Example 2.2. Considering the Theorem 2.1, we can find several solutions of (3) as numerical examples.

For $m = -1$, $(x_{1_0}, x_{2_0}, x_{3_0}) = (x_1, x_2, x_3) = (126, 124, 5)$

For $m = 0$, $(x_{1_1}, x_{2_1}, x_{3_1}) = (18387054, 18055756, 1909)$

For $m = 1$, $(x_{1_2}, x_{2_2}, x_{3_2}) = (2854294786854, 2802866051956, 752141)$

... ...

Corollary 2.3. There are some relations among sequences of integer solutions of (3) as the following:

(i) $x_{1_{m+1}} + x_{2_{m+1}} = 10x_{3\,m+1}^2$, for $m = -1, 0, 1, 2, \ldots$
(ii) $22x_{2_{m+1}} - 109x_{3_{m+1}}^2 = 3$, for $m = -1, 0, 1, 2, \ldots$
(iii) $11x_{2_{m+1}} - 109x_{1_{m+1}} = 30$, for $m = -1, 0, 1, 2, \ldots$
(iv) $111x_{3_{m+1}}^2 - 22x_{1_{m+1}} = 3$, for $m = -1, 0, 1, 2, \ldots$

Proof. It is easily to seen that all conditions are satisfied for $m = -1$ and $m = 0$. Also, it can be provedby using Mathematical induction and computer program for $m > 0$.

Corollary 2.4. Each of the following statements is represented by quartic (bi-quadratic) integers.

(i) $25x_{3_{m+1}}^4 - x_{1_{m+1}}.x_{2_{m+1}} = \mathcal{A}^4$, for $m = -1, 0, 1, 2, \ldots$ and $\mathcal{A} \in \mathbb{Z}$.
(ii) $\frac{1}{2}\left[x_{1_{m+1}}^2 + x_{2_{m+1}}^2 - 50x_{3_{m+1}}^4\right] = \mathcal{B}^4$, for $m = -1, 0, 1, 2, \ldots$ and $\mathcal{B} \in \mathbb{Z}$.

Proof. We can see that It can be proved byDefinition 1.1 and mathematical induction as well as computer program.

Corollary 2.5. Sequences of general non-zero integer solutions of (3) are written in the terms of polygonal number of rank $x_{3_{m+1}}$ with size 22 as follows:

$$x_{1_{m+1}} + x_{2_{m+1}} - 9x_{3_{m+1}} = t_{22, x_{3_{m+1}}}$$

Proof. It can be proved using Definition 1.3, Mathematical induction and also computer program.

Corollary 2.6. Following expressions give relations between sequences of general non-zero integer solutions of (3) and the terms of pyramidal number of rank $x_{3_{m+1}}$ with size 5 or rank $(x_{3_{m+1}} - 1)$ with size 3.

(i) $(x_{1_{m+1}} + x_{2_{m+1}}).(x_{3_{m+1}} + 1) = 20\, P^5_{x_{3_{m+1}}}$, for $m = -1, 0, 1, 2, ...$

(ii) $(x_{1_{m+1}} + x_{2_{m+1}} - 10).x_{3_{m+1}} = 60 P^3_{x_{3_{m+1}-1}}$, for $m = -1, 0, 1, 2, ...$

Proof. Using Definition 1.4, Mathematical induction and computer program, we can prove the Corollary 2.6.

Corollary 2.7. Pyramidal number of rank $x_{3_{m+1}}$ with size 4 and polygonal number of rank $x_{3_{m+1}}$ with size 3 is written by sequences of general non-zero integer solutions of (3) as follows:

$$(x_{1_{m+1}} + x_{2_{m+1}} - 10).x_{3_{m+1}} = 30(P^4_{x_{3_{m+1}}} - t_{3,x_{3_{m+1}}})$$

for $m = -1, 0, 1, 2, ...$

Proof. Considering Definition 1.4, Definition 1.3, Mathematical induction and computer program, Corollary 2.7 can be demonstrated for $m = -1, 0, 1, 2, ...$

Corollary 2.8.$3(x_{1_{m+1}} - x_{2_{m+1}})$ is a nasty number for $m = -1, 0, 1, 2, ...$

Proof. In a similar way of the proofs of above corollaries, for $m = -1, 0, 1, 2, ...$, it is proved by Definition 1.2, Mathematical induction and computer program.

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Some Non-Extendible Regular Triple P_s- Sets

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ABSTRACT

Many open problems in Number Theory has been waiting to solve for a long time before. One of them is Diophantine 3-tuples $\boldsymbol{P_s}$ which is defined as "sets with the property such that product of any two distinct elements adding $\boldsymbol{s}$ integer is a square integer".
The purpose of this study is to determine some special non-extendible regular P_s Diophantine 3-tuples for a fixed integer s. To get them, solutions of diophantine equations are considered. Some characteristic properties are determined for such sets. Results are demonstrated using some notions such as quadratic reciprocity law, legendre symbols, quadratic residues, modular arithmetic and so on … from algebraic and elementary number theory.

Keywords: Diophantine Triples; Diophantine Equations; Integral Solutions; Quadratic Reciprocity Theorem; Legendre Symbol; Modular Arithmetic; Pell Equations.
2010 AMS Mathematics Subject Classification: 11A07, 11D09, 11A15.

1. INTRODUCTION AND PRELIMINARIES

The purpose of this brief paper is to determine some specific non-extendible regular Diophantine triples with propert $\boldsymbol{P_s}$ for fixed integer $\boldsymbol{s = 41}$ or $\boldsymbol{s = -41}$. To prove those sets are not extendible, we consider quadratic diophantine equations and apply factorization method of integers on them. Then, we determine their congruences types and regularity. Also, we classify the elements of Diophantine sets with property $\boldsymbol{P_s}$ for fixed integer $\boldsymbol{s = 41}$ or $\boldsymbol{s = -41}$ using basic concepts of elementary and algebraic number theories. The paper will constitute the basis for our next paper.
All of the references [1-17] are significant and handy for the topic of this paper. Following basic concepts and theories are used to get our main results for the paper.

Definition 1.1. (Diophantine Sets, Diophantine Triples with Property $\boldsymbol{P_s}$) For any integer s, a Diophantine $\boldsymbol{P_s}$-set with n-tuples is defined as the following:

A set $\{\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_n}\}$ is n-tuple of different positive integers where $\boldsymbol{\theta_i \theta_j + s}$ is always a perfect square of an integer for every distinct i and j, where $\boldsymbol{i, j = 1, 2, \dots, n}$.

As a particular case, the set is called $\boldsymbol{P_s}$- Diophantine triple if $n = 3$.

Definition 1.2. (Regular Diophantine Triple) If $\boldsymbol{P_s}$- Diophantine triple $\{\boldsymbol{\rho, \sigma, \tau}\}$ satisfies the condition

$$(\boldsymbol{\tau - \sigma - \rho})^2 = \boldsymbol{4(\rho.\sigma + s)}\quad(\boldsymbol{1})$$

it is called Regular Diophantine Triple.

Definition 1.3. (Quadratic Residue) If $u \in \mathbb{N}$ and $\gamma \in \mathbb{Z}$ with $\gcd(\gamma, u) = 1$, then γ is to be a quadratic residue modulo u if there exists an integer y such that

$$y^2 \equiv \gamma \ (mod\ u) \qquad (2)$$

Besides, if (2) has no solution, then γ is called a non-quadratic residue modulo u.

Definition 1.4.(Legendre Symbol)If $m \in \mathbb{Z}$ and $q > 2$ is a prime, then

$$\left(\frac{m}{q}\right) = \begin{cases} 1, & \text{if } a \text{ is quadratic residue } modulo\ q \\ 0, & \text{if } q|m \\ -1, & \text{otherwise} \end{cases} \tag{3}$$

and $\left(\frac{m}{q}\right)$ is called the Legendre Symbol of m with respect to q.

Theorem 1.5. (The Quadratic Reciprocity Law)Letp, q be distinct odd primes.Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \tag{4}$$

where $\left(\frac{\cdot}{\cdot}\right)$ represents Legendre symbol. Also, Quadratic Residuacity of 2 modulo q is given by

$$\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8} \tag{5}$$

and also Quadratic Residuacity of (-1) modulo q is defined by

$$\left(\frac{-1}{q}\right) = \begin{cases} 1 & \text{if } q \equiv 1 (\text{mod } 4) \\ -1 & \text{if } q \equiv -1 (\text{mod } 4) \end{cases} \tag{6}$$

Definition 1.6. (Congruence Type)If the elements of set$\boldsymbol{P_s}$- Diophantine triples are reduced modulo 4, it is called congruence type column and represented by [..., ..., ...].

2. MAIN RESULTS

Theorem 2.1. Let$\mathcal{A} = \{2, 160, 200\}$ be a set with three positive integers. Then following statements are satisfied.

(i) $\mathcal{A} = \{2, 160, 200\}$ is a non-extendible to Diophantine quadruple with property P_{+41}.

(ii)$\{2, 160, 200\}$ is regular Diophantine triple with property P_{+41} and congruence type column of the set is $[2, 0, 0]$.

Proof. **(i)**Let $\{2, 160, 200\}$can be extended to Diophantine quadruple with property P_{+41}. Then, $\{2, 160, 200, \jmath\}$is Diophantine quadruple for any positive integer $\jmath$.Then, there exist u_1, u_2, u_3 integers such that following equations are hold.

$$2\jmath + 41 = {u_1}^2 \tag{7}$$

$$160\jmath + 41 = {u_2}^2 \tag{8}$$

$$200\jmath + 41 = {u_3}^2 \tag{9}$$

Simplify$\jmath$ between (7) and (9), we obtain

$$100{u_1}^2 - {u_3}^2 = 4059 \tag{10}$$

By factorizing the both side of (10), then we get following table:

Table 1:Integer solutions of $100u_1^2 - u_3^2 = 4059$

Solutions	1.Class of Solutions	2.Class of Solutions	3.Class of Solutions	4.Class of Solutions
u_1	∓ 203	∓ 23	∓ 19	∓ 7
u_3	∓ 2029	∓ 221	∓ 179	∓ 29

Dropping of $\jmath$ between (7) and (8), we get

$$80u_1^2 - u_2^2 = 3239 \qquad (11)$$

From the values of variables in the Table 1, we calculate $u_1^2 = 41209$, $u_1^2 = 529$, $u_1^2 = 361$ and $u_1^2 = 49$ respectively. Putting values ofu_1^2 into the (11), $u_2^2 = 3293481$, $u_2^2 = 39081$, $u_2^2 = 25641$ and $u_2^2 = 681$ are obtained. This is a contradiction sinceu_2's values are not integer solutions of (11).
So, there is not any such $\jmath \in \mathbb{Z}^+$ and the $P_{+41} = \{2, 160, 200\}$ can not be extended to Diophantine quadruple.

(ii) Let consider regularity condition (1) in Definition 1.2. Then, it is easily seen that $P_{+41} = \{2, 160, 200\}$is a regular Diophantine triple.

We can see that the congruence type column of $\mathcal{A} = \{2, 160, 200\}$ is [2, 0, 0]. Also, one of the congruence type of [12] is obtained from (ii) in Theorem 2.1.

Theorem 2.2. Let$\mathcal{B} = \{2, 200, 244\}$ be a set of three positive integers. The following expressions are hold.

(i) $\mathcal{B} = \{2, 200, 244\}$cannotextendible to Diophantine quadruple with property P_{+41}.
(ii) $\{2, 200, 244\}$ is a regular Diophantine triple with property P_{+41} and congruence type column of the set is $[2, 0, 0]$.

Proof. The proof of the Theorem 2.2 is obtained as like the proof of the Theorem 2.1.

Theorem 2.3. If a set$\mathcal{C} = \{4,62,100\}$ is of three positive integers, then the following statements are provided.

(i) $\mathcal{C} = \{4,62,100\}$ can non-extendible to Diophantine quadruple with property P_{+41}.
(ii) $\mathcal{C} = \{4,62,100\}$is a regular Diophantine triple with property P_{+41} and congruence type column of the set is $[0, 2, 0]$.

Proof.(i)Given that $\mathcal{C} = \{4, 62, 100, \mathfrak{K}\}$ be a Diophantine quadruplewith property P_{+41} for $\mathfrak{K} \in \mathbb{Z}^+$.Considering the Definition 1.1, we have

$$4\,\mathfrak{K} + 41 = \mathfrak{v}_1^2 \qquad (12)$$

$$62\,\mathfrak{K} + 41 = \mathfrak{v}_2^2 \qquad (13)$$

$$100\,\mathfrak{K} + 41 = \mathfrak{v}_3^2 \qquad (14)$$

for $\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3 \in \mathbb{Z}$. Dropping $\mathfrak{K}$ from (12) and (14), following equation is obtained;

$$100\mathfrak{v}_1^2 - 4\mathfrak{v}_3^2 = 3936 \qquad (15)$$

And also in a same way, we obtain following equation from (12) and (13);

$$31\mathfrak{v}_1{}^2 - 2\mathfrak{v}_2{}^2 = 1189 \qquad (16)$$

From the factorization method in the set of integers, we have following table for $(\mathfrak{v}_1, \mathfrak{v}_3)$ soutions in the set of integers.

Table 2: Integer solutions of $100\mathfrak{v}_1{}^2 - 4\mathfrak{v}_3{}^2 = 3936$

Solutions	1.Class of Solutions	2.Class of Solutions
$(\mathfrak{v}_1, \mathfrak{v}_3)$	$(\mp 25, \mp 121)$	$(\mp 17, \mp 79)$

Using the $\mathfrak{v}_1$ values from Table 2 and substituting in the (16), we get$\mathfrak{v}_2{}^2 = 9093$ or $\mathfrak{v}_2{}^2 = 3885$. This shows that $\mathfrak{v}_2$ is not integer solution for (16). It is a contradiction and $P_{+41} = \{4, 62, 100\}$ is a Diophantine triple.

(ii)Applying the condition (1) of Definition 1.2 on $\mathcal{C} = \{4,62,100\}$, we can see that the set is regular triple. Besides, using modulo 4 on the set , we obtain congruence type column as $[0, 2, 0]$ which is not found in [12].

Theorem 2.4. A set$\mathcal{D} = \{4, 100, 146\}$is of three positive integers. $\mathcal{D} = \{4, 100, 146\}$ can be non-extended to Diophantine quadruple with property P_{+41}. Also, $\mathcal{D} = \{4, 100, 146\}$ is regular and congruence type column of the set is $[0, 0, 2]$.

Proof. The proof of the Theorem 2.4 is obtained in the similar way of the Theorem 2.1. or Theorem 2.3. Applying (mod 4) on the set , congruence type column is given by $[0, 0, 2]$ which is not determined in [12].

Theorem 2.5. Given that $\mathcal{E} = \{8, 10, 40\}$ is a set of positive integers. Then, $\mathcal{E} = \{8, 10, 40\}$ can not be extended to Diophantine quadruple with property P_{+41}. Besides, $\mathcal{E} = \{8, 10, 40\}$ is regular Diophantine triple and alsocongruence type column of the set is given by $[0, 2, 0]$.

Proof. The proof of the Theorem 2.5 is obtained in the similar way of the Theorem 2.1. or Theorem 2.3. Congruence type column is determined by $[0, 2, 0]$ as like in [12].

Theorem 2.6. Let $\mathcal{F} = \{10, 40, 92\}$ is a set of positive integers. Thus, both $\mathcal{F} = \{10, 40, 92\}$can not extendible to Diophantine quadruple with property P_{+41} and $\mathcal{F} = \{10, 40, 92\}$is regular Diophantine triple. Additionally, $[0, 0, 2]$is congruence type column of the$\mathcal{F}$ set.

Proof. The proof of the Theorem 2.6 is obtained in a same way of the Theorem 2.1. or Theorem 2.3. As we said in the proof of Theorem 2.4,congruence type column is defined by $[0, 0, 2]$not in [12].

Remark. New sets for P_{+41} Diophantine triples can be found with our method and all of them can be generalized in the terms of some special numbers or special integer sequences.

Theorem 2.7. Following conditions satisfy for Diophantine sets with property P_{+41}:

(i) $\mathfrak{X} \in \mathbb{Z}^+$,$\mathfrak{X}$ is divided by 3 or any multiplies of 3, then $\mathfrak{X} \notin P_{+41}$.

(ii) ($\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 7 or any multiplies of 7, then $\mathfrak{X} \notin P_{+41}$) or ($\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 11 or any multiplies of 11, then $\mathfrak{X} \notin P_{+41}$) or ($\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 13 or any multiplies of 13, then $\mathfrak{X} \notin P_{+41}$) or ($\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 17 or any multiplies of 17, then $\mathfrak{X} \notin P_{+41}$) or ($\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 19 or any multiplies of 19, then $\mathfrak{X} \notin P_{+41}$) or ($\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 29 or any multiplies of 29, then $\mathfrak{X} \notin P_{+41}$),… so on…

Proof.i)Given that both $\mathscr{b} \in \mathbb{Z}^+$ and also $\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{F}$ is divided by 3 or any multiplies of 3, be elements of P_{+41} Diophantine set.From the Definition 1.1, we get

$$3\mathfrak{X}\mathscr{b} + 41 = \mathbb{X}^2 \qquad (17)$$

for an integer$\mathfrak{X}$ and$\mathbb{X}$. Applying (mod 3) on the both side of (17), we obtain

$$\mathbb{X}^2 \equiv 2\ (mod\ 3) \qquad (18)$$

From (5) of Theoem 1.1, we have

$$\left(\frac{2}{3}\right) = -1 \qquad (19)$$

It implies that (18) doesn't have solution and it is a contradiction. So, If $\mathfrak{X} \in \mathbb{Z}^+$, $\mathfrak{X}$ is divided by 3 or any multiplies of 3, then $\mathfrak{X} \notin P_{+41}$.

ii)The first satisfied condition of Theorem 2.7 is proved by using Theorem 1.1. In a similar way and using Definition 1.3, Definition 1.4, Definition 1.5 and Theorem 1.1, others can be proved.

Remark. Theorem 2.7 can be extended for some integers.

Theorem 2.8. Let$\mathcal{G} = \{7, 30, 63\}$be a set of three positive integers. Then, following expressions are provided with property P_{-41}.

(i) $\mathcal{G} = \{7, 30, 63\}$cannot extendible to Diophantine quadruple with property P_{-41}.
(ii) $\mathcal{G} = \{7, 30, 63\}$ is a regular Diophantine triple with property P_{-41} and congruence type column of the set is determined by $[3, 2, 3]$.

Proof.(i)Assume that $\mathcal{G} = \{7, 30, 63, \mathfrak{T}\}$ be a Diophantine quadruple with property P_{-41} for $\mathfrak{T} \in \mathbb{Z}^+$.Applying Definition 1.1 on the $\mathcal{G}$ set, we get

$$7\mathfrak{T} - 41 = \mathfrak{w}_1{}^2 \qquad (20)$$

$$30\mathfrak{T} - 41 = \mathfrak{w}_2{}^2 \qquad (21)$$

$$63\mathfrak{T} - 41 = \mathfrak{w}_3{}^2 \qquad (22)$$

for $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \in \mathbb{Z}$. Using simplification of $\mathfrak{T}$ from (20) and (22);

$$\mathfrak{w}_3{}^2 - 9\mathfrak{w}_1{}^2 = 328 \qquad (23)$$

is obtained.In a same vein, we have following equation from (2.14) and (2.15);

$$7\mathfrak{w}_2{}^2 - 30\mathfrak{w}_1{}^2 = 943 \qquad (24)$$

We have following table from the equation (23):

Table 3: Integer solutions of $\mathfrak{w}_3{}^2 - 9\mathfrak{w}_1{}^2 = 328$

Solutions	1.Class of Solutions	2.Class of Solutions
$(\mathfrak{w}_3, \mathfrak{w}_1)$	$(\mp 83, \mp 27)$	$(\mp 43, \mp 13)$

We obtain$\mathfrak{w}_2{}^2 = 3259$ or $\mathfrak{w}_2{}^2 = 859$ by considering Table3. It is clear that $\mathfrak{w}_2$ is not integer solution of the (2.18) equation. So, it is a contradiction. $P_{-41} = \{7, 30, 63\}$is a Diophantine triple and can not extendible to Diophantine quadruple with property P_{-41}.

ii) We can easily see that the set$\{7, 30, 63\}$ is regular triple from the condition (1) of Definition 1.2. Also,practicing (mod 4), we get congruence type column as like$[3, 2, 3]$which is not in [12].

Theorem 2.9. For a set$\mathcal{H} = \{9, 18, 49\}$ includes three positive integers, the following statements are provided.

(i) $\mathcal{H} = \{9, 18, 49\}$cannot extendible to Diophantine quadruple with property P_{-41}.
(ii) $\mathcal{H} = \{9, 18, 49\}$is a regular Diophantine triple with property P_{-41} and congruence type column of the set is given by $[1, 2, 1]$.

Proof. The proof of the Theorem 2.9 is got in the same way of the proof of Theorem 2.8. From modular algorithm, we have congruence type column as $[1, 2, 1]$which is not in [12].

Theorem 2.10. For a set$\mathcal{J} = \{9, 49, 98\}$contains three positive integers then $\mathcal{J} = \{9, 49, 98\}$ can not be extented to Diophantine quadruple with property P_{-41}. Besides, $\mathcal{J} = \{9, 49, 98\}$ is a regular Diophantine triple set and congruence type column of the set is determined by $[1, 1, 2]$.

Proof. The proof of the Theorem 2.10 is hadlike the proof of Theorem 2.8. We have congruence type column as like $[1, 1, 2]$given in [12].

Theorem 2.11: There is no Diophantine set P_{-41} contains any multiple of 4, 13, 17, 23, 29, 31...so on...

Proof. Suppose that g is an element of Diophantine set P_{-41} . If $4\mathfrak{z}$ is also an element of set P_{-41} for $\mathfrak{z} \in Z$, then

$$4\mathfrak{z}g - 41 = \mathbb{y}^2 \qquad (25)$$

is satisfied for some integer $\mathbb{y}$. Considering (mod 4) and apply on the (2.19), we get

$$\mathbb{y}^2 \equiv 3 \ (mod\ 4) \qquad (26)$$

If $\mathbb{y}$ is odd integer then we have $\mathbb{y}^2 \equiv 1 \ (mod\ 4)$ and also $\mathbb{y}^2 \equiv 0 \ (mod\ 4)$is obtained otherwise. So, (26) can not be solved. This is a contradiction. Thus, there is no Diophantine set P_{-41} contains any multiple of 4.

Remark. There are lots of integers which aren't in Diophantine set$P_{-41}$and one may determine others using our method based on preliminaries section.

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HYERS-ULAM INSTABILITY OF LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT

In this paper we have obtained integral sufficient conditions under which the zero solution of a nonlinear differential equation of second order with initial condition is unstable in the sense of Hyers and Ulam . We also have proved the Hyers -Ulam Instability of a linear differential equation of second order with initial condition. To illustrate the results we have given an example.

Keywords: Hyers -Ulam , Instability, Nonlinear ,Linear, Differential equations.

1. INTRODUCTION

In [10], Ulam posed the basic problem of the stability of functional equations: Give conditions in order for a linear mapping near an approximately linear mapping to exist . This problem was partially solved by Hyers in 1941, for approximately additive mappings on Banach spaces [3]. In 1978 Rassias in his work [8], has generalized that result obtained by Hyers.

After then, many mathematicians have extensively investigated the stability problems of functional Equations. More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations.

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza [6] and Alsina [1].

This result of Obloza has been generalized by authors [4,5,9,11]. Qarawani [7] investigated the Hyers-Ulam stability nonlinear differential equation of second order $y'' + y = h(x)y^{\beta}$ with the initial conditions $y(x_0) = 0 = y'(x_0)$. In [2] Brillouë t-Belluot indicated that there are only few outcomes of which we could say that they concern nonstability of functional equations.

In this paper, we investigate for the first time the Hyers-Ulam instability of the following linear differential equation of second order

$$y'' + y = \alpha(x)y \tag{1.1}$$

with the initial conditions

$$y(x_0) = 0 = y'(x_0) \tag{1.2}$$

Moreover we have proved the Hyers-Ulam instability of the nonlinear differential equation of second order

$$y'' + y = h(x)y^{\beta} \tag{1.3}$$

with the initial conditions

$$y(x_0) = 0 = y'(x_0) \tag{1.4}$$

where $\alpha(x)$ is a function defined in R, $h \in C^1(I), I = [x_0, x] \subseteq \mathbb{R}, x_0 > 0,$ and β is a ratio of two positive odd integers.

2. PRELIMINARIES

Definition 2.1 We will say that the Eq. (1.1) has the Hyers -Ulam stability with the initial conditions (1.2) if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0, y \in C^2(I)$ where x is sufficiently large in $\mathbb{R}$, if

$$| y'' + y - \alpha(x)y | \leq \varepsilon \tag{2.1}$$

then there exists some solution $w_0 \in C^2(I)$ of the Eq. (1.1), such that $| y(x) - w_0(x) | \leq K\varepsilon$ and satisfies the initial conditions

$$w_0(x_0) = 0 = w_0'(x_0)$$

Definition 2.2 We will say that Eq. (1.3) has the Hyers-Ulam stability with initial conditions (1.4) if there exists a positive constant $K > 0$ with the following property:
For every $\varepsilon > 0,\ y \in C^2(I)$ where x is sufficiently large in $\mathbb{R}$, if

$$| y'' + y - h(x)y^{\beta} | \leq \varepsilon \tag{2.2}$$

then there exists some solution $w \in C^2(I)$ of the Eq. (1.3) and

$$w(x_0) = w'(x_0) = 0$$

such that $| y(x) - w_0(x) | \leq K\varepsilon.$

3. MAIN RESULTS ON HYERS-ULAM INSTABILITY

Theorem 3.1 Suppose that $y \in C^2(I)$ and $|y'(x)| \leq |y(x)|$ for all $x \geq x_0$, such that satisfies the inequality

$$| y'' + y - \alpha(x) \quad y | \leq \varepsilon$$

with the initial condition

$$y(x_0) = 0 = y'(x_0)$$

If $\int_{x_0}^{\infty} \alpha(t)dt$ diverges, then the zero solution of Eq. (1.1) is unstable in the sense of Hyers and Ulam.

Proof. Suppose that $y \in C^2(I)$ satisfies the inequality (2.1) with the initial conditions (1.2). We will show that zero solution $w_0(x) \equiv 0$ of the Eq. (1) will satisfy the inequality $|y(x) - w_0(x)| > k\varepsilon$. On the contrary, let us assume that there exists $\varepsilon > 0$ such that $\sup_{x \geq x_0} |y(x) - w_0(x)| \leq k\varepsilon.$ Then we can find a constant $M > 0$ such that

$$M = \sup_{x \geq x_0} |y(x)|.$$

From the inequality (2.1) we have

$$-\varepsilon \leq y'' + y - \alpha(x)y \leq \varepsilon \tag{3.1}$$

Multiply the inequality (3.1) by $' \geq 0$ and then integrate we obtain

$$-2\varepsilon y \leq y'^{\,2}(x) + y^2(x) - 2\int_{x_0}^{x} \alpha(t) \quad yy'dt \leq 2\varepsilon y \tag{3.2}$$

Since $|y'(x)| \leq |y(x)|$, then from (3.2) we get

$$2y^2(x) \geq -2\varepsilon y + 2\int_{x_0}^{x} \alpha(t)yy'dt \geq -2\varepsilon M + 2|y(x_*)y'(x_*)| \int_{x_0}^{x} \alpha(t) \quad dt$$

Therefore

$$M^2 \geq -\varepsilon M + y'^2(x_*) \int_{x_0}^{x} \alpha(t)\ dt = \infty$$

Similarly if we multiply the inequality (3.1) by $y' \le 0$, then we get

$$2y^2(x) \ge 2\varepsilon y + 2\int_{x_0}^{x}\alpha(t)yy'dt \ge -2\varepsilon M + 2|y(x_*)y'(x_*)|\int_{x_0}^{x}\alpha(t) \quad dt$$

and

$$M^2 \ge -\varepsilon M + y'^2(x_*)\int_{x_0}^{x}\alpha(t) \quad dt = \infty$$

This contradicts the hypothesis that M is a constant.
Thus, we have $\sup_{x\ge x_0}|y(x)| > k\varepsilon$. Obviously, $w_0(x) = 0$ satisfies the Eq. (1.1) and the zero initial condition (1.2) such that $\sup_{x\ge x_0}|y(x) - w_0(x)| > k\varepsilon$. Therefore the Eq. (1.1) is instable in the sense of Hyers and Ulam.

Example 3.1 Consider the equation

$$y''(x) + y(x) = e^x y \tag{3.3}$$

with the initial condition

$$y(x_0) = 0 = y'(x_0) \tag{3.4}$$

We will show that zero solution $w_0(x) \equiv 0$ of the equation (3.3) will satisfy the inequality $|y(x) - w_0(x)| > k\varepsilon$. On the contrary, suppose that there exists $\varepsilon > 0$ such that $\sup_{x\ge x_0}|y(x) - w_0(x)| \le k\varepsilon$. Then we can find a constant $M > 0$ such that

$$M = \sup_{x\ge x_0}|y(x)|.$$

Multiply the following inequality by $y' \ge 0$ and then integrate

$$-\varepsilon \le y'' + y - e^x y \le \varepsilon$$

we obtain

$$-2\varepsilon y \le y'^{\,2}(x) + y^2(x) - 2\int_{x_0}^{x}e^x yy'dt \le 2\varepsilon y$$

If we assume that $|y'(x)| \le |y(x)|$ for all $x \ge x_0$, then we get

$$^2 \quad \ge -2\varepsilon y + 2\int_{x_0}^{x}e^x yy'dt \ge -2\varepsilon y + 2y(x_*)y'(x_*)\int_{x_0}^{x}e^t dt$$

Since the integral $\int_{x_0}^{\infty} e^t dt$ diverges, then for $x \to \infty$, we get

$$M^2 = \infty.$$

Similarly if we multiply the inequality (3.3) by $y' \le 0$, then we get

$$2y^2(x) \ge 2\varepsilon y + 2\int_{x_0}^{x}\alpha(t) \quad yy'dt \ge 2\varepsilon y + 2y(x_*)y'(x_*)\int_{x_0}^{x}e^t dt$$

and for sufficiently large x we obtain

$$^2 \ge \varepsilon M + 2y(x_*)y'(x_*)\int_{x_0}^{x}e^t dt = \infty$$

Obviously, $w_0(x) = 0$ satisfies the equation (3.3) and the zero initial condition (3.4) such that $\sup\limits_{x \geq x_0} |y(x) - w_0(x)| > k\varepsilon$. Therefore the equation (3.3) is unstable in the sense of Hyers and Ulam.

Theorem 3.2 Suppose that $y \in C^2(I)$ and $|y'(x)| \leq |y(x)|$ for all $x \geq x_0$, such that satisfies the inequality

$$| \ y'' + y - \ h(x) \quad y^{\alpha} \ | \leq \varepsilon \tag{3.5}$$

with the initial condition

$$y(x_0) = 0 = y'(x_0) \tag{3.6}$$

If $\int\limits_{x_0}^{\infty} h(t)dt$ diverges, then the Eq. (1.3) is unstable in the sense of Hyers and Ulam.

Proof. On the contrary, suppose that there exists $\varepsilon > 0$ such that $\sup\limits_{x \geq x_0} |y(x) - w_0(x)| \leq k\varepsilon$. Then we can find a constant $M > 0$ such that

$$M = \sup_{x \geq x_0} |y(x)|.$$

From the inequality (3.5) we have

$$-\varepsilon \leq y'' + y - \ h(x) \quad y^{\beta} \ \leq \varepsilon \tag{3.7}$$

Multiply the inequality (3.7) by $y' \geq 0$ and then integrate we obtain

$$-2\varepsilon y \leq y'^{\,2}(x) + y^2(x) - 2\int\limits_{x_0}^{x} h(x) \quad y^{\beta} y' dt \leq 2\varepsilon y$$

From which we get that

$$2y^2(x) \geq -2\varepsilon y + 2\int\limits_{x_0}^{x} h(t) \quad y^{\beta} y' dt = -2\varepsilon M + 2y^{\beta}(x_*) y'(x_*) \int\limits_{x_0}^{x} \alpha(t) dt$$

Since the integral $\int\limits_{x_0}^{\infty} \alpha(t)dt$ diverges, then for $x \longrightarrow \infty$, we get

$$M^2 \geq -\varepsilon M + y'^{\beta+1}(x_*) \int\limits_{x_0}^{x} \alpha(t) \quad dt = \infty$$

Similarly if we multiply the inequality (3.7) by $y' \leq 0$, then we get

$$2y^2(x) \geq 2\varepsilon y + 2\int\limits_{x_0}^{x} \alpha(t) y^{\beta} y' dt \geq -2\varepsilon M + 2\left|y^{\beta}(x_*) y'(x_*)\right| \int\limits_{x_0}^{x} \alpha(t) dt, \text{ for any}$$

$$x \geq x_0 .$$

And for sufficiently large x we obtain

$$M^2 \geq -\varepsilon M + \left|y'^{\beta+1}(x_*)\right| \int\limits_{x_0}^{x} \alpha(t) \quad dt = \infty$$

So we conclude that $\sup\limits_{x \geq x_0} |y(x)| > k\varepsilon$. Obviously, $w_0(x) = 0$ satisfies the Eq. (1.3) and the zero initial condition (1.4) such that $\sup\limits_{x \geq x_0} |y(x) - w_0(x)| > k\varepsilon$. Therefore the Eq. (1.3) is unstable in the sense of Hyers and Ulam.

Example 3.2 Consider the Eq.

$$y''(x) + y(x) = \frac{y^{3/2}}{x+1} \tag{3.8}$$

with the initial condition

$$y(x_0) = 0 = y'(x_0) \tag{3.9}$$

We will show that zero solution $w_0(x) \equiv 0$ of the Eq. (3.8)will satisfy the inequality $|y(x) - w_0(x)| > k\varepsilon.$ On the contrary, suppose that there exists $\varepsilon > 0$ such that $\sup_{x \geq x_0} |y(x) - w_0(x)| \leq k\varepsilon.$ Then we can find a constant $M > 0$ such that

$$M = \sup_{x \geq x_0} |y(x)|.$$

Multiply the following inequality by $y' \geq 0$ and then integrate

$$-\varepsilon \leq y'' + y - \frac{y^{3/2}}{x+1} \leq \varepsilon \tag{3.10}$$

we obtain

$$-2\varepsilon y \leq y'^{\,2}(x) + y^2(x) - 2\int_{x_0}^{x} \frac{y^{3/2}}{t+1} y' dt \leq 2\varepsilon y$$

If we assume that $|y'(x)| \leq |y(x)|$ for all $x \geq x_0$, then we get

$$2y^2(x) \geq -2\varepsilon y + 2\int_{x_0}^{x} \frac{y^{3/2}}{t+1} y' dt \geq -2\varepsilon y + 2y^{3/2}(x_*)y'(x_*)\int_{x_0}^{x} \frac{dt}{t+1}$$

Since the integral $\int_{x_0}^{\infty} \frac{dt}{t+1}$ diverges, then for $x \to \infty$, we get

$$M^2 \geq -\varepsilon M + y'^{5/2}(x_*)\int_{x_0}^{x} \frac{dt}{t+1} = \infty$$

Similarly if we multiply the inequality (3.10) by $y' \leq 0$, then for sufficiently large x we obtain

$$M^2 \geq -\varepsilon M + 2\left|y'^{5/2}(x_*)\right|\int_{x_0}^{x} \frac{dt}{t+1} = \infty$$

So we conclude that $\sup_{x \geq x_0} |y(x)| > k\varepsilon.$ Obviously, $w_0(x) = 0$ satisfies the Eq. (3.8)and the zero initial condition (3.9) such that $\sup_{x \geq x_0} |y(x) - w_0(x)| > k\varepsilon.$ Therefore the Eq. (3.8)is unstable in the sense of Hyers and Ulam.

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INTRODUCTION TO Q-NEUTROSOPHIC SOFT FIELDS

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ABSTRACT

The objective of the current work is to extend the thought of Q-neutrosophic soft sets to fields. In this paper, we define the notion of Q-neutrosophic soft fields. Structural characteristics of it are investigated.

Keywords:neutrosophic soft field; Q-neutrosophic soft field; Q-neutrosophic soft set

1. INTRODUCTION

Fuzzy sets were established by Zadeh [20] as a tool to deal with uncertain data. Smarandache [16] initiated the neutrosophic idea as a new extension of the fuzzy set. A neutrosophic set (NS) [15] is a mathematical notion serving issues containing imprecise, indeterminate, and inconsistent data. In [11], Molodtsovintroduced the soft sets as another way to handle uncertainty. Since its initiation, a plenty of hybrid models of soft sets have been produced,for example, fuzzy soft sets [14], neutrosophic soft sets (NSSs) [9]. NSSs were extended to Q-neutrosophic soft sets (Q-NSSs) [3] a new model that deals with two-dimensional uncertain data. Q-NSSs were further investigated and their basic operations, relations and measures of entropy distance and similarity were discussed in [1-3].Different hybrid models of fuzzy sets and soft sets were utilized in different branches of mathematics, including algebra. This was started by Rosenfeld in 1971 [14] when he established the idea of fuzzy subgroup. Since then, the theories and approaches of fuzzy soft sets on different algebraic structures developed rapidly. In this respect, severalauthors have utilizeddistinct hybrid models of fuzzy sets to differentdomains of algebra such as groups, fields, rings semigroups and BCK/BCI-algebras [4,5,8,12,19]. NSs and NSSs have received moreattention in studying the algebraic structures of set theories dealing with uncertainty. Bera and Mahapatra introduced the notion of neutrosophic soft groups [6], neutrosophic soft fields [7]. Moreover, two-dimensional hybrid models of fuzzy sets and soft sets were also applied to different algebraic structures. Solairaju and Nagarajan [17] presentedQ-fuzzy groups. Also, Rasuli [13] defined Q-fuzzy subrings and anti Q-fuzzy subrings, while Thiruveni and Solairaju introduced neutrosophic Q-fuzzy subgroups [18].Inspired by the above works and to utilize Q-NSSs to different algebraic structures, in the current paper, we define the notion of Q-neutrosophic soft fields (Q-NSFs) and discuss some of its structural characteristics.

2. PRELIMINARIES

Here, we recall the basic definitions related to this work.

Definition 2.1. [3]LetX be a universal set,Q be a nonempty set and$A \subseteq E$be a set of parameters. Let$\mu^l QNS(X)$be the set of all multi Q-NSs on$X$with dimension $l = 1$. A pair $(\Gamma_Q, A)$is called a Q-NSS over$X$, where$\Gamma_Q: A \to \mu^l QNS(X)$is a mapping, such that$\Gamma_Q(e) = \phi \text{ if } e \notin A$.

Definition 2.2. [1]The union of two Q-NSSs$(\Gamma_Q, A)$and$(\Psi_Q, B)$is the Q-NSS(Λ_Q, C)written as$(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$,where$C = A \cup B$and for all$c \in C, (x, q) \in X \times Q$, the truth-membership, indeterminacy-membership and falsity-membership of $(\Lambda_Q, C)$are as follows:

$$T_{\Lambda_{Q(c)}}(x,q)=\begin{cases}T_{\Gamma_{Q(c)}}(x,q) & \text{if } c\in A-B,\\ T_{\Psi_{Q(c)}}(x,q) & \text{if } c\in B-A,\\ \max\{T_{\Gamma_{Q(c)}}(x,q),T_{\Psi_{Q(c)}}(x,q)\} & \text{if } c\in A\cap B,\end{cases}$$

$$I_{\Lambda_{Q(c)}}(x,q)=\begin{cases}I_{\Gamma_{Q(c)}}(x,q) & \text{if } c\in A-B,\\ I_{\Psi_{Q(c)}}(x,q) & \text{if } c\in B-A,\\ \min\{I_{\Gamma_{Q(c)}}(x,q),I_{\Psi_{Q(c)}}(x,q)\} & \text{if } c\in A\cap B,\end{cases}$$

$$F_{\Lambda_{Q(c)}}(x,q)=\begin{cases}F_{\Gamma_{Q(c)}}(x,q) & \text{if } c\in A-B,\\ F_{\Psi_{Q(c)}}(x,q) & \text{if } c\in B-A,\\ \min\{F_{\Gamma_{Q(c)}}(x,q),F_{\Psi_{Q(c)}}(x,q)\} & \text{if } c\in A\cap B.\end{cases}$$

Definition 2.3. **[1]**The intersection of two Q-NSSs$(\Gamma_Q,A)$and$(\Psi_Q,B)$is the Q-NSS(Λ_Q,C)written as$(\Gamma_Q,A)\cap(\Psi_Q,B)=(\Lambda_Q,C)$, where$C=A\cap B$and for all$c\in Cand(x,q)\in X\times \mathrm{Q}$ the truth-membership, indeterminacy-membership and falsity-membership of$(\Lambda_Q,C)$are as follows:

$$T_{\Lambda_{Q(c)}}(x,q)=\min\{T_{\Gamma_{Q(c)}}(x,q),T_{\Psi_{Q(c)}}(x,q)\},$$
$$I_{\Lambda_{Q(c)}}(x,q)=\max\{I_{\Gamma_{Q(c)}}(x,q),I_{\Psi_{Q(c)}}(x,q)\},$$
$$F_{\Lambda_{Q(c)}}(x,q)=\max\{F_{\Gamma_{Q(c)}}(x,q),F_{\Psi_{Q(c)}}(x,q)\}.$$

3. Q-NEUTROSOPHIC SOFT FIELDS

In the current section, we present Q-NSFs and discuss several related properties.

Definition 3.1. Let (Γ_Q,A) be a Q-NSS over a field $(F,+,.)$. Then (Γ_Q,A) is said to be a Q-NSF over $(F,+,.)$ if for all $e\in A$, $\Gamma_Q(e)$ is a Q-neutrosophic subfield of $(F,+,.)$, where $\Gamma_Q(e)$ is a mapping given by $\Gamma_Q(e): F\times Q\to[0,1]^3$.

Definition 3.2. Let$(F,+,.)$be a field and(Γ_Q,A)be a Q-NSS over$(F,+,.)$. Then,$(\Gamma_Q,A)$is called a Q-NSF over$(F,+,.)$if for all$x,y\in F,q\in Q$and$e\in A$it satisfies:

(1) $T_{\Gamma_{Q(e)}}(x+y,q)\geq\min\left\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(y,q)\right\}, I_{\Gamma_{Q(e)}}(x+y,q)\leq \max\left\{I_{\Gamma_{Q(e)}}(x,q),\ I_{\Gamma_{Q(e)}}(y,q)\right\}, F_{\Gamma_{Q(e)}}(x+y,q)\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(y,q)\}$.

(2) $T_{\Gamma_{Q(e)}}(-x,q)\geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(-x,q)\leq I_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(-x,q)\leq F_{\Gamma_{Q(e)}}(x,q)$.

(3) $T_{\Gamma_{Q(e)}}(x.y,q)\geq\min\left\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(y,q)\right\}, I_{\Gamma_{Q(e)}}(x.y,q)\leq\max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\right\}, F_{\Gamma_{Q(e)}}(x.y,q)\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(y,q)\}$.

(4) $T_{\Gamma_{Q(e)}}(x^{-1},q)\geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x^{-1},q)\leq I_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(x^{-1},q)\leq F_{\Gamma_{Q(e)}}(x,q)$.

Example 3.3. Let$F=(\mathbb{R},+,.)$be the field of real numbers and A=$\mathbb{N}$ the set of natural numbers be the parametric set. Define a Q-NSS(Γ_Q,A)as follows for$q\in Q,x\in\mathbb{R}$and$m\in\mathbb{N}$

$$T_{\Gamma_{Q(m)}}(x,q)=\begin{cases}0 & \text{if } x \text{ is rational}\\ \dfrac{1}{9m} & \text{if } x \text{ is irrational,}\end{cases}$$

$$I_{\Gamma_{Q(m)}}(x,q)=\begin{cases}1-\dfrac{1}{3m} & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational,}\end{cases}$$

$$F_{\Gamma_{Q(m)}}(x,q) = \begin{cases} 1+\dfrac{3}{m} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

It is clear that $(\Gamma_Q, \mathbb{N})$ is a Q-NSF over F.

Proposition 3.4. *Let (Γ_Q, A) be a Q-NSF over $(F,+,.)$. Then, for the additive identity 0_F and the multiplicative identity 1_F, for all $x \in F, q \in Q$ and $e \in A$ the following hold*

(1) $T_{\Gamma_{Q(e)}}(0_F,q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(0_F,q) \leq I_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(0_F,q) \leq F_{\Gamma_{Q(e)}}(x,q)$.

(2) $T_{\Gamma_{Q(e)}}(1_F,q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(1_F,q) \leq I_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(1_F,q) \leq F_{\Gamma_{Q(e)}}(x,q)$, for $x \neq 0_F$.

(3) $T_{\Gamma_{Q(e)}}(0_F,q) \geq T_{\Gamma_{Q(e)}}(1_F,q), I_{\Gamma_{Q(e)}}(0_F,q) \leq I_{\Gamma_{Q(e)}}(1_F,q), F_{\Gamma_{Q(e)}}(0_F,q) \leq F_{\Gamma_{Q(e)}}(1_F,q)$.

Proof. $\forall x \in F, q \in Q$ and $e \in A$

(1) $T_{\Gamma_{Q(e)}}(0_F,q) = T_{\Gamma_{Q(e)}}(x-x,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(x,q)\} = T_{\Gamma_{Q(e)}}(x,q)$,

$I_{\Gamma_{Q(e)}}(0_F,q) = I_{\Gamma_{Q(e)}}(x-x,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x,q)\} = I_{\Gamma_{Q(e)}}(x,q)$,

$F_{\Gamma_{Q(e)}}(0_F,q) = F_{\Gamma_{Q(e)}}(x-x,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(x,q)\} = F_{\Gamma_{Q(e)}}(x,q)$.

(2) $T_{\Gamma_{Q(e)}}(1_F,q) = T_{\Gamma_{Q(e)}}(x.x^{-1},q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(x,q)\right\} = T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(1_F,q) = I_{\Gamma_{Q(e)}}(x.x^{-1},q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x,q)\right\} = I_{\Gamma_{Q(e)}}(x,q),\ F_{\Gamma_{Q(e)}}(1_F,q) = F_{\Gamma_{Q(e)}}(x.x^{-1},q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(x,q)\right\} = F_{\Gamma_{Q(e)}}(x,q)$.

(3) Follows directly by applying 1. □

Theorem 3.5. *A Q-NSS (Γ_Q, A) over the field $(F,+,.)$ is a Q-NSF if and only if for all $x, y \in F, q \in Q$ and $e \in A$*

(1) $T_{\Gamma_{Q(e)}}(x-y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\},\ F_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.

(2) $T_{\Gamma_{Q(e)}}(x.y^{-1},q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x.y^{-1},q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}, F_{\Gamma_{Q(e)}}(x.y^{-1},q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.

Proof. Suppose that (Γ_Q, A) is a Q-NSF over $(F,+,.)$. Then,

$$T_{\Gamma_{Q(e)}}(x-y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(-y,q)\} \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\},$$
$$I_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(-y,q)\} \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\},$$
$$F_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(-y,q)\} \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}.$$

Also,

$$T_{\Gamma_{Q(e)}}(x.y^{-1},q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y^{-1},q)\} \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\},$$
$$I_{\Gamma_{Q(e)}}(x.y^{-1},q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y^{-1},q)\} \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\},$$
$$F_{\Gamma_{Q(e)}}(x.y^{-1},q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y^{-1},q)\} \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}.$$

Conversely, Suppose that conditions 1 and 2 are satisfied. We show that for each $e \in A$, (Γ_Q, A) is a Q-neutrosophic subfield

$$T_{\Gamma_{Q(e)}}(-x,q) = T_{\Gamma_{Q(e)}}(0_F - x,q) \geq \min\{T_{\Gamma_{Q(e)}}(0_F,q), T_{\Gamma_{Q(e)}}(x,q)\}$$
$$\geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(x,q)\} = T_{\Gamma_{Q(e)}}(x,q),$$
$$I_{\Gamma_{Q(e)}}(-x,q) = I_{\Gamma_{Q(e)}}(0_F - x,q) \leq \max\{I_{\Gamma_{Q(e)}}(0_F,q), I_{\Gamma_{Q(e)}}(x,q)\}$$
$$\leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x,q)\right\} = I_{\Gamma_{Q(e)}}(x,q),$$
$$F_{\Gamma_{Q(e)}}(-x,q) =_{\Gamma_{Q(e)}} (0_F - x,q) \leq max\{F_{\Gamma_{Q(e)}}(0_F,q), F_{\Gamma_{Q(e)}}(x,q)\}$$

$$\leq max\{F_{\Gamma_Q(e)}(x,q), F_{\Gamma_Q(e)}(x,q)\} = F_{\Gamma_Q(e)}(x,q)\}$$

also,

$$T_{\Gamma_Q(e)}(x+y,q) = T_{\Gamma_Q(e)}(x-(-y),q) \geq \min\{T_{\Gamma_Q(e)}(x,q), T_{\Gamma_Q(e)}(y,q)\},$$
$$I_{\Gamma_Q(e)}(x+y,q) = I_{\Gamma_Q(e)}(x-(-y),q) \leq \max\{I_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(y,q)\},$$
$$F_{\Gamma_Q(e)}(x+y,q) = F_{\Gamma_Q(e)}(x-(-y),q) \leq \max\{F_{\Gamma_Q(e)}(x,q), F_{\Gamma_Q(e)}(y,q)\}.$$

Next,

$$T_{\Gamma_Q(e)}(x^{-1},q) = T_{\Gamma_Q(e)}(1_F.x^{-1},q) \geq \min\{T_{\Gamma_Q(e)}(1_F,q), T_{\Gamma_Q(e)}(x,q)\}$$
$$\geq \min\{T_{\Gamma_Q(e)}(x,q), T_{\Gamma_Q(e)}(x,q)\} = T_{\Gamma_Q(e)}(x,q),$$
$$I_{\Gamma_Q(e)}(x^{-1},q) = I_{\Gamma_Q(e)}(1_F.x^{-1},q) \leq \max\{I_{\Gamma_Q(e)}(1_F,q), I_{\Gamma_Q(e)}(x,q)\}$$
$$\leq \max\{I_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(x,q)\} = I_{\Gamma_Q(e)}(x,q),$$
$$F_{\Gamma_Q(e)}(x^{-1},q) =_{\Gamma_Q(e)} (1_F.x^{-1},q) \leq \max\{F_{\Gamma_Q(e)}(1_F,q), F_{\Gamma_Q(e)}(x,q)\}$$
$$\leq \max\{F_{\Gamma_Q(e)}(x,q), F_{\Gamma_Q(e)}(x,q)\} = F_{\Gamma_Q(e)}(x,q)\}$$

and

$$T_{\Gamma_Q(e)}(x.y,q) = T_{\Gamma_Q(e)}(x(y^{-1})^{-1},q) \geq \min\{T_{\Gamma_Q(e)}(x,q), T_{\Gamma_Q(e)}(y,q)\},$$
$$I_{\Gamma_Q(e)}(x.y,q) = I_{\Gamma_Q(e)}(x(y^{-1})^{-1},q) \leq \max\{I_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(y,q)\},$$
$$F_{\Gamma_Q(e)}(x.y,q) = F_{\Gamma_Q(e)}(x(y^{-1})^{-1},q) \leq \max\{F_{\Gamma_Q(e)}(x,q), F_{\Gamma_Q(e)}(y,q)\}.$$

This completes the proof. □

Theorem 3.6._Let_ (Γ_Q, A) _and_ (Ψ_Q, B) _be two Q-NSFs over_ $(F, +, .)$. _Then,_ $(\Gamma_Q, A) \cap (\Psi_Q, B)$ _is also Q-NSF over_ $(F, +, .)$.

Proof. Let $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, A \cap B)$. Now, $\forall x, y \in F, q \in Q$ and $e \in A \cap B$,

$$T_{\Lambda_Q(e)}(x-y,q) = \min\{T_{\Gamma_Q(e)}(x-y,q), T_{\Psi_Q(e)}(x-y,q)\}$$
$$\geq \min\{\min\{T_{\Gamma_Q(e)}(x,q), T_{\Gamma_Q(e)}(y,q)\}, \min\{T_{\Psi_Q(e)}(x,q), T_{\Psi_Q(e)}(y,q)\}\}$$
$$= \min\{\min\{T_{\Gamma_Q(e)}(x,q), T_{\Psi_Q(e)}(x,q)\}, \min\{T_{\Gamma_Q(e)}(y,q), T_{\Psi_Q(e)}(y,q)\}\}$$
$$= \min\{T_{\Lambda_Q(e)}(x,q), T_{\Lambda_Q(e)}(y,q)\},$$

also,

$$I_{\Lambda_Q(e)}(x-y,q) = \max\{I_{\Gamma_Q(e)}(x-y,q), I_{\Psi_Q(e)}(x-y,q)\}$$
$$\leq \max\{\max\{I_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(y,q)\}, \max\{I_{\Psi_Q(e)}(x,q), I_{\Psi_Q(e)}(y,q)\}\}$$
$$= \max\{\max\{I_{\Gamma_Q(e)}(x,q), I_{\Psi_Q(e)}(x,q)\}, \max\{I_{\Gamma_Q(e)}(y,q), I_{\Psi_Q(e)}(y,q)\}\}$$
$$= \max\{I_{\Lambda_Q(e)}(x,q), I_{\Lambda_Q(e)}(y,q)\},$$

similarly, $F_{\Lambda_Q(e)}(x-y,q) \leq \max\{F_{\Lambda_Q(e)}(x,q), F_{\Lambda_Q(e)}(y,q)\}$. Next,

$$T_{\Lambda_Q(e)}(x.y^{-1},q) = \min\{T_{\Gamma_Q(e)}(x.y^{-1},q), T_{\Psi_Q(e)}(x.y^{-1},q)\}$$
$$\geq \min\{\min\{T_{\Gamma_Q(e)}(x,q), T_{\Gamma_Q(e)}(y,q)\}, \min\{T_{\Psi_Q(e)}(x,q), T_{\Psi_Q(e)}(y,q)\}\}$$
$$= \min\{\min\{T_{\Gamma_Q(e)}(x,q), T_{\Psi_Q(e)}(x,q)\}, \min\{T_{\Gamma_Q(e)}(y,q), T_{\Psi_Q(e)}(y,q)\}\}$$
$$= \min\{T_{\Lambda_Q(e)}(x,q), T_{\Lambda_Q(e)}(y,q)\},$$

also,

$$I_{\Lambda_Q(e)}(x.y^{-1},q) = \max\{I_{\Gamma_Q(e)}(x.y^{-1},q), I_{\Psi_Q(e)}(x.y^{-1},q)\}$$
$$\leq \max\{\max\{I_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(y,q)\}, \max\{I_{\Psi_Q(e)}(x,q), I_{\Psi_Q(e)}(y,q)\}\}$$
$$= \max\{\max\{I_{\Gamma_Q(e)}(x,q), I_{\Psi_Q(e)}(x,q)\}, \max\{I_{\Gamma_Q(e)}(y,q), I_{\Psi_Q(e)}(y,q)\}\}$$
$$= \max\{I_{\Lambda_Q(e)}(x,q), I_{\Lambda_Q(e)}(y,q)\}$$

similarly, we can show $F_{\Lambda_Q(e)}(x.y^{-1},q) \leq \max\{F_{\Lambda_Q(e)}(x,q), F_{\Lambda_Q(e)}(y,q)\}$. This completes the proof.

Remark 3.7. For two Q-NSFs$(\Gamma_Q, A)$and(Ψ_Q, B)over$(F, +, .)$, $(\Gamma_Q, A) \cup (\Psi_Q, B)$is not generally a Q-NSF.

For example, let $F = (\mathbb{Q}, +, .)$, $E = 2\mathbb{Z}$. Consider two Q-NSFs (Γ_Q, E) and (Ψ_Q, E) over F as follows: for $x \in \mathbb{Q}, q \in Q$ and $m \in \mathbb{Z}$

$$T_{\Gamma_{Q(4m)}}(x,q)=\begin{cases}0.50 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0 & otherwise,\end{cases}$$
$$I_{\Gamma_{Q(4m)}}(x,q)=\begin{cases}0 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0.25 & otherwise,\end{cases}$$
$$F_{\Gamma_{Q(4m)}}(x,q)=\begin{cases}0.40 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0.10 & otherwise,\end{cases}$$
and
$$T_{\Psi_{Q(4m)}}(x,q)=\begin{cases}0.70 & \text{if } x=6tm, \exists t\in\mathbb{Z},\\ 0 & otherwise,\end{cases}$$
$$I_{\Psi_{Q(4m)}}(x,q)=\begin{cases}0 & \text{if } x=6tm, \exists t\in\mathbb{Z},\\ 0.50 & otherwise,\end{cases}$$
$$F_{\Psi_{Q(4m)}}(x,q)=\begin{cases}0.20 & \text{if } x=6tm, \exists t\in\mathbb{Z},\\ 0.40 & otherwise.\end{cases}$$
Let $(\Gamma_Q,A)\cup(\Psi_Q,B)=(\Lambda_Q,E)$. For $m=2, x=8, y=12$ we have
$$T_{\Lambda_{Q(8)}}(8-12,q)=T_{\Lambda_{Q(8)}}(-4,q)=\max\{T_{\Gamma_{Q(8)}}(-4,q),T_{\Psi_{Q(8)}}(-4,q)\}=\max\{0,0\}=0$$
and
$$\begin{aligned}&\min\{T_{\Lambda_{Q(8)}}(8,q),T_{\Lambda_{Q(8)}}(12,q)\}\\ &=\min\{\max\{T_{\Gamma_{Q(8)}}(8,q),T_{\Psi_{Q(8)}}(8,q)\},\max\{T_{\Gamma_{Q(8)}}(12,q),T_{\Psi_{Q(8)}}(12,q)\}\}\\ &=\min\{\max\{0.50,0\},\max\{0,0.7\}\}\\ &=\min\{0.50,0.70\}=0.50.\end{aligned}$$
Hence, $T_{\Lambda_{Q(8)}}(8-12,q)<min\{T_{\Lambda_{Q(8)}}(8,q),T_{\Lambda_{Q(8)}}(12,q)\}$. Thus, the union is not a Q-NSF.

6. Conclusion

We have introduced the concept of Q-neutrosophic soft fields. We have investigated some of its structural characteristics

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SOLVING THE BEAM DEFLECTION PROBLEM USING AL-TEMEME TRANSFORMS

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ABSTRACT

In this paper, an enhancement to the beam deflection problem is performed through the substitution of $q(x)$ by $\frac{1}{x^4}$, this substitution is performed to reduce the beam load intensity, also the enhanced beam deflection problem is solved using two new transforms, which are complex AL-Tememe and AL-Tememetransforms. the results (solutions) from complex AL-Tememe and AL-Tememe transforms are compares to each other, both transforms have the ability to solve the enhanced problem of the beam deflection.

Keywords: Complex AL-Tememe transform; AL-Tememe transform; deflection of the beam, differential equations; famous function; Inverse of AL-Tememe transform; Inverse of complex AL-Tememe transform; uniform distributed load.

6. INTRODUCTION

The beam deflection problem is widely discussed in many books [7-11], where many methods are used to solve that problem, however the use of Al-Tememe and complex Al-Tememe transforms never discussed before. AL-Tememe and complex AL-Tememe are two transforms that emerged at 2016 and 2018 respectively, these transforms can solve some types of deferential equations, which can be used in many scientific fields, such as physics, engineering and bio-medical signal processing [2,4,5,6]. In this paper, the problem of deflection of beam is solved using complex AL-Tememe and AL-Tememe transforms, and the solutions from these transforms are compared.

7. BASIC CONCEPTS

It is necessary to mention some relevant definitions, functions, proprieties and theorems to make the calculations clearer.

2.1Definition of complex AL-Tememe transform [2]:

A complex AL-Tememe transform for the function $f(x)$, $x > 1$ is defined by the integral: $T^c[f(x)] = \int_1^\infty x^{-ip} f(x)dx = F(ip)$.
Such that this integral is convergent in $[1, \infty]$, p is a positive constant, and x^{-ip} is the kernel of this transform and $i = \sqrt{-1}$.

2.2 Definition of inverse complexAL-Tememe transform [2]:

If $T^c[f(x)] = F(ip)$ represents a complex AL-Tememe transform of$f(x)$, then $f(x)$ is said to be the inverse the AL-Tememe transform and it can be written by: $f(x) = T^{c^{-1}}\big(F(ip)\big)$.

2.3 Propriety of complex AL-Tememe transform [2]:

A complex AL-Tememe transform linear: $T^c(Af(x) \pm BT^c\big(g(x)\big) = AT^c(f(x)) \pm BT^c\big(g(x)\big)$. Where A and B are constants, the function $f(x)$ and $g(x)$ are defined when $x > 1$.

2.4 Complex AL-Tememe transform of some famous function [2] :

1. $T^c(1) = \frac{1}{-1+ip}$
2. $T^c(x^n) = \frac{1}{ip-(n+1)}$, $n \in R$

2.5 Inverse of complex AL-Tememe transform of famous function [2]:

1) $T^{c^{-1}}\left(\frac{1}{-1+ip}\right) = 1.$
2) $T^{c^{-1}}\left(\frac{1}{ip-(n+1)}\right) = x^n$, $n \in R.$
3) $T^{c^{-1}}\left(\frac{1}{(ip-1)^2}\right) = \ln(x).$

2.6 Theorem [2]:

Let $y(x)$ be defined function for $x > 1$, and its derivatives $y'(x), y''(x), \cdots, y^n(x)$ exist, then: $T^c[x^n y^n(x)] = -y^{(n-1)}(1) - (ip-n)y^{(n-2)}(1) - \cdots - (ip-n)\big(ip - (n-1)\big).\cdots.(ip-2)y(1) + (ip-n)\big(ip-(n-1)\big)\cdots(ip-1)F(ip) \qquad n \in Z^+.$

2.7 Definition of AL-Tememe transform [1]:

Al-Tememe Transform for the function $f(x)$; $x > 1$ is defined by the following integral $T[f(x)] = \int_1^\infty x^{-p} f(x)dx = F(p)$. Such that this integral is convergent in some region, p is a positive constant, and x^{-p} the kernel of Al-Tememe Transform.

2.8 Definition of inverse AL-Tememe transform [1]:

Let $f(x)$ be a function where $x > 1$ and $T[f(x)] = F(p)$, $f(x)$ is said to be an inverse for Al-Tememe Transform and written as:$T^{-1}[F(p)] = f(x)$, where T^{-1} returns the transform to the original function.

2.9 Propriety of AL-Tememe transform [1]:

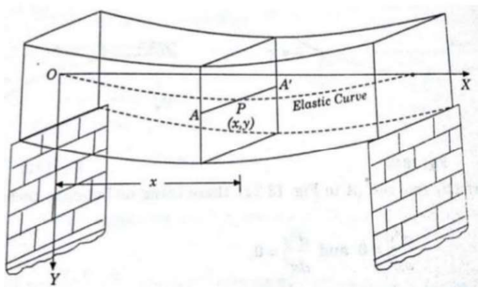
The transformation is characterized by the linear propriety, that is: $T[Af(x) \pm Bg(x)] = AT[f(x)] \pm BT[g(x)]$ where A and B are constants, the functions $f(x)$ and $g(x)$ are defined when $x > 1$.

<u>2.10 Table of selected Al-Tememe transforms [1]</u>

Function $f(x)$	$F(p) = \int_1^\infty x^{-p} f(x)dx$	Region of convergence
$k, k = Constant$	$\frac{k}{p-1}$	$p > 1$
$x^a, a \in R$	$\frac{1}{p-(a+1)}$	$p > a+1$
$\ln x$	$\frac{1}{(p-1)^2}$	$p > 1$
$x^a \ln x, a \in R$	$\frac{1}{[p-(a+1)]^2}$	$p > a+1$

3. DEFLECTION OF THE BEAM PROBLEM[3]:

- If a beam of length L with rectangular cross section and homogenous elastic material (e.g. steel) is considered as shown in figure (1).
- And if a load is applied to the beam in vertical plane through the axis of symmetry (the x-axis), the beam is going to bent.
- If a cross-section of the beam cutting the elastic curve in p and the neutral surface in the line AA'.



(a) Figure (1)

- Then the bending moment M about AA' is given by Bernoulli- Euler law.
  $M = \frac{EI}{R}(3.1)$
  Where:
  $E =$ modulus of electricity of the beam.
  $I =$ moment of inertia of the cross-section AA'.
  $R =$ radios of curvature of the elastic curve at $p(x, y)$.
  If the deformation of the beam is small, the slope of the elastic curve is also small so that it is possible to neglect $(\frac{dy}{dx})^2$ in the formula $R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$.
- For small defection, $= \frac{1}{\frac{d^2y}{dx^2}}$.
- Hence, (3.1) bending moment $M = EI\frac{d^2y}{dx^2}$.
- Shear force $= \frac{dM}{dx} = EI\frac{d^3y}{dx^3}$.
- Intensity of loading$= \frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = q(x)$.
- The sum of moments about any section due to external forces on the left of the section, if anti-clock is taken as positive and if clockwise is taken as negative.
- The most important supports corresponding boundary conditions are:

1) Simply supported as shown in figure (2):

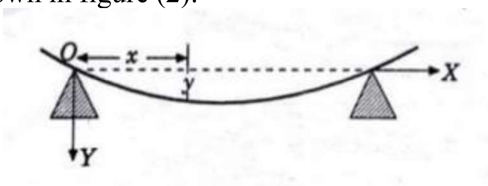


Figure (2)

- No deflection and bending moment exist. Then:
  $y(0) = 0, y''(0) = 0$.
  $y(l) = 0, y''(l) = 0$.

2) Completed at$x = 0$, free at $x = l$ as shown in figure (3).

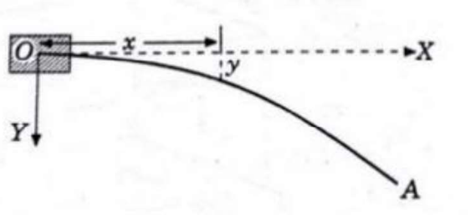


Figure (3)

- At $x = 0$, the deflection and slop of the beam being both zero. At $x = 1$, there are no bending moment and shear force. We have, $y(0) = y'(0) = 0,\ y''(l) = y'''(l) = 0$.

3) Clamped at both ends: The defection and the slop of the beam being both zero, then:
$y(0) = 0,\ y'(0) = 0$.
$y(l) = 0, y'(l) = 0$.

4. THE DEFLECTION OF A BEAM CARRYING UNIFORM DISTRIBUTED LOAD

Assume that a uniform loaded beam of length L is supported at both ends, as shown in figure (4). The deflection $y(x)$ is a function of horizontal position x, it is given by the differential equation: $\frac{d^4y}{dx^4} = \frac{1}{EI} q(x)(4.1)$

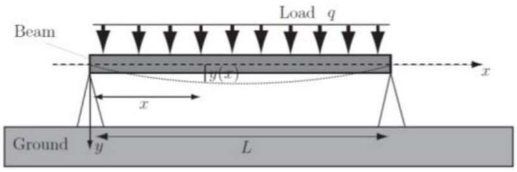


Figure (4)

Where $q(x)$ is the load per unit length at point x. it is assumed in this problem that $q(x) = q$ (q is a constant).

The boundary conditions are:

(i). No deflection at $x = 0$ and $x = l$.

(ii). No bending moment of the beam at $x = 0$ and $x = l$.

$\left.\begin{matrix} y(0)=0 \\ y(l)=0 \end{matrix}\right\} no\ deflection\ at\ x = 0\ and\ x = l$

$\left.\begin{matrix} y''(0)=0 \\ y''(l)=0 \end{matrix}\right\} no\ bending\ moment\ at\ x = 0\ and\ x = l$

4.1 Solving the deflection of a beam carrying uniform distributed load using complex AL-Tememe transform

Complex AL-Tememe transform is used to solve the problem of deflection for a beam that carrying a uniform distributed load. After substituting each $q(x)$ by $\frac{1}{x^4}$ equation (4.1) becomes: $x^4y^{(4)} - \frac{1}{EI} = 0 \quad y(1) = 0, y''(1) = 0$

By taking a complex AL-Tememe transform to both sides:

$T^c(x^4y^{(4)}) - T^c\left(\frac{1}{EL}\right) = 0,$

$-y'''(1) - (ip-4)y''(1) - (ip-4)(ip-3)y'(1) - (ip-4)(ip-3)(ip-2)y(1) + (ip-4)(ip-3)(ip-2)(ip-1)T^c(y) - \frac{1}{EI}T^c(1) = 0$.

$T^c(y) = \frac{y'''(1)}{(ip-4)(ip-3)(ip-2)(ip-1)} + \frac{y'(1)}{(ip-2)(ip-1)} + \frac{1}{EL} \cdot \frac{1}{(ip-4)(ip-3)(ip-2)(ip-1)^2}.$

By taking the inverse of a complex AL-Tememe transform to both sides:

$y = T^{c^{-1}}\left[\frac{y'''(1)}{(ip-4)(ip-3)(ip-2)(ip-1)}\right] + T^{c^{-1}}\left[\frac{y'(1)}{(ip-2)(ip-1)}\right] + \frac{1}{EL} . T^{c^{-1}}\left[\frac{1}{(ip-4)(ip-3)(ip-2)(ip-1)^2}\right]$.

Now, we take

$$\frac{1}{(ip-4)(ip-3)(ip-2)(ip-1)} = \frac{A}{ip-4} + \frac{B}{ip-3} + \frac{C}{ip-2} + \frac{D}{ip-1}$$

After simple computations, we get:

$A = \frac{1}{6}, B = -\frac{1}{2}, C = \frac{1}{2}, D = -\frac{1}{6}.$

Then

$T^{c^{-1}}\left[\frac{1}{(ip-4)(ip-3)(ip-2)(ip-1)}\right] = T^{c^{-1}}\left(\frac{\frac{1}{6}}{ip-4}\right) + T^{c^{-1}}\left(-\frac{\frac{1}{2}}{ip-3}\right) + T^{c^{-1}}\left(\frac{\frac{1}{2}}{ip-2}\right) + T^{c^{-1}}\left(-\frac{\frac{1}{6}}{ip-1}\right) = \left[\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{6}\right]y'''.$

Also, we take

$\frac{1}{(ip-2)(ip-1)} = \frac{A}{ip-2} + \frac{B}{ip-1}$.

After simple computations:

$A = 1, and\ B = -1$.

Then:

$T^{c^{-1}}\left[\frac{1}{(ip-2)(ip-1)}\right] = T^{c^{-1}}\left[\frac{1}{ip-2}\right] + T^{c^{-1}}\left[\frac{-1}{ip-1}\right] = (x-1)y'$.

As well as, we take

$\frac{1}{(ip-4)(ip-3)(ip-2)(ip-1)^2} = \frac{A}{ip-4} + \frac{B}{ip-3} + \frac{C}{ip-2} + \frac{D}{ip-1} + \frac{E}{(ip-1)^2}$.

After, simple computations:

$A = \frac{1}{18}, B = -\frac{1}{4}, C = \frac{1}{2}, D = -\frac{11}{36}, E = -\frac{1}{6}$.

Then:

$T^{c^{-1}}\left[\frac{\frac{1}{EI}}{(ip-4)(ip-3)(ip-2)(ip-1)^2}\right] = T^{c^{-1}}\left[\frac{\frac{1}{18}}{ip-4}\right] + T^{c^{-1}}\left[\frac{-\frac{1}{4}}{ip-3}\right] + T^{c^{-1}}\left[\frac{\frac{1}{2}}{ip-2}\right] +$
$T^{c^{-1}}\left[\frac{-\frac{11}{36}}{ip-1}\right] + T^{c^{-1}}\left[\frac{-\frac{1}{6}}{(ip-1)^2}\right] = \left(\frac{1}{18}x^3 - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{11}{36} - \frac{1}{6}\ln(x)\right).\frac{1}{EI}$.

Then:

$y = \left(\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{6}\right)y'''(1) + (x-1)y'(1) + \left(\frac{1}{18}x^3 - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{11}{36} - \frac{1}{6}\ln(x)\right).\frac{1}{EI}$ (4.2).

To use the boundary condition$y''(l) = 0$, and by taking the second derivative of (4.2) then:

$y(x) = \frac{1}{24EI}x^4 - \frac{l}{EI}x^3 + \frac{l^3}{24EI}x$ (4.3).

The above equation gives the deflection of the beam at a distance x.

To find the maximum deflection, put $x = \frac{l}{2}$ in equation (4.3).

4.2 Solving the deflection of a beam carrying uniform distributed load using AL-Tememe transform

AL-Tememe transform is used to solve the problem of deflection for a beam that carrying a uniform distributed load. After substituting each $q(x)$ by $\frac{1}{x^4}$ equation (4.1) becomes:

$x^4y^{(4)} - \frac{1}{EI} = 0 \quad y(1) = 0, y''(1) = 0$

By taking AL-Tememe transform to both sides:

$T(x^4y^{(4)}) - T\left(\frac{1}{EL}\right) = 0,$

$-y'''(1) - (p-4)y''(1) - (p-4)(p-3)y'(1) - (p-4)(p-3)(p-2)y(1) +$
$(p-4)(p-3)(p-2)(p-1)T(y) - \frac{1}{EI}T(1) = 0$.

$T(y) = \frac{y'''(1)}{(p-4)(p-3)(p-2)(p-1)} + \frac{y'(1)}{(p-2)(p-1)} + \frac{1}{EL}.\frac{1}{(p-4)(p-3)(p-2)(p-1)^2}$.

By taking the inverse of AL-Tememe transform to both sides:

$y = T^{-1}\left[\frac{y'''(1)}{(p-4)(p-3)(p-2)(p-1)}\right] + T^{-1}\left[\frac{y'(1)}{(p-2)(p-1)}\right] + \frac{1}{EL}.T^{-1}\left[\frac{1}{(p-4)(p-3)(p-2)(p-1)^2}\right]$.

Now, we take

$\frac{1}{(p-4)(p-3)(p-2)(p-1)} = \frac{A}{p-4} + \frac{B}{p-3} + \frac{C}{p-2} + \frac{D}{p-1}$.

After simple computations, we get:

$A = \frac{1}{6}\ , B = -\frac{1}{2}\ , C = \frac{1}{2}\ , D = -\frac{1}{6}$.

Then

$T^{-1}\left[\frac{1}{(p-4)(p-3)(p-2)(p-1)}\right] = T^{-1}\left(\frac{\frac{1}{6}}{p-4}\right) + T^{-1}\left(-\frac{\frac{1}{2}}{p-3}\right) + T^{-1}\left(\frac{\frac{1}{2}}{p-2}\right) + T^{-1}\left(-\frac{\frac{1}{6}}{p-1}\right)$.

Also, we take

$\frac{1}{(p-2)(p-1)} = \frac{A}{p-2} + \frac{B}{p-1}$.

After simple computations, we have:

$A = 1, and\ B = -1$.

As well as, we take:

$\frac{1}{(p-4)(p-3)(p-2)(p-1)^2} = \frac{A}{p-4} + \frac{B}{p-3} + \frac{C}{p-2} + \frac{D}{p-1} + \frac{E}{(p-1)^2}$.

After, simple computations, we have:

$A = \frac{1}{18}, B = -\frac{1}{4}, C = \frac{1}{2}, D = -\frac{11}{36}, E = -\frac{1}{6}$.

Now:

$T^{-1}\left[\frac{y'''(1)}{(p-4)(p-3)(p-2)(p-1)^2}\right] = (\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{6})y'''$.

Also, $T^{-1}\left[\frac{y'(1)}{(p-2)(p-1)}\right] = (x-1)y'$.

And, $\frac{1}{EI}T^{-1}\left[\frac{1}{(p-4)(p-3)(p-2)(p-1)^2}\right] = [\frac{1}{8}x^3 - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{11}{36} - \frac{1}{6}\ln x]\frac{1}{EI}$.

Finally, $y = \left[\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{6}\right]y''' + (x-1)y' + [\frac{1}{8}x^3 - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{11}{36} - \frac{1}{6}\ln x]\frac{1}{EI}$.

5. Conclusions

There are many solutions to the beam deflection problem, however Al-Tememe transforms (Al-Tememe and Complex Al-Tememe) are never used before to solve this problem. The previous computations solved the beam deflection problem through the reduction of load that provided over the beam, by dividing the beam deflection equation by $x^4$to became $y^{(4)} = \frac{1}{x^4}\frac{1}{EI}$. Both transforms gave the same results therefore it is possible to use either of them to solve the beam deflection problem.

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FABER POLYNOMIAL COEFFICIENT BOUNDS OF THE MEROMORPHIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH JACKSON'S(p,q) −DERIVATIVE

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ABSTRACT

In this article, we introduce a new subclass of meromorphic bi-univalent functions, using $(p,q)-Jackson$ derivative. We obtain the general coefficient estimates $|a_m|$ for such functions belonging to this subclass and examine their early coefficient bounds by applying Faber polynomial coefficient expansions.

Keywords: Analytic functions, Meromorphic functions, Bi-univalent functions, Faber polynomial, q-calculus.

1. INTRODUCTION

We start by letting $\Omega = \{z: z \in \mathbb{R} \text{ and } 1 < |z| < \infty\}$, and Σ be the class of meromorphic functions of the form

$$h(z) = z + a_0 + \sum_{m=1}^{\infty} \frac{a_m}{z^m}. \tag{1}$$

that are univalent in Ω. Its well known that every function$h \in \Sigma$ has an inverse h^{-1} defined by

$$\begin{cases} h^{-1}(h(z)) = z, \ z \in \Omega \\ h(h^{-1}(\omega)) = \omega, \ \mu < |\omega| < \infty, \mu > 0 \end{cases}.$$

For a brief history in the class Σ, you can see [2,4,12,14]. A univalent function in $\Omega$is said to be bi-univalent if its inverse map is also univalent there. The function$h \in \Sigma$ is said to be bi-univalent and meromorphic if $h^{-1} \in \Sigma$. The family of these functions is denoted by Σ_M. Springer [14] proved $|a_3| \leq 1, |a_3 + \frac{1}{2}{a_1}^2| \leq \frac{1}{2}$ and conjectured that $|a_{2m-1}| \leq \frac{(2m-1)!}{m!(m-1)!}$ for $(m = 1,2,\dots)$. The bounds for general coefficients $|a_m|$ of meromorphic bi-univalent functions were obtained by Hamidi et al. [3] and they examined their early coefficient bounds.

The Faber Polynomial expansion of the inverse map of $h \in \Sigma$ of the form (1),

$$\varphi = h^{-1} = \omega - b_0 - \frac{b_1}{\omega} - \frac{b_1 b_0 + b_2}{\omega^2} - \frac{{b_1}^2 + b_1 {b_0}^2 + 2b_0 b_2 + b_3}{\omega^3} + \cdots = \omega - b_m - \sum_{m\geq 1}^{\infty} \frac{1}{m} K_{m+1}^m \frac{1}{\omega^m}, \ \omega \in \Omega. \tag{2}$$

where

$$K_{m+1}^m = m{b_0}^{m-1} b_1 + m(m-1){b_0}^{m-2} b_2 + \frac{1}{2} m(m-1)(m-2){b_0}^{m-3}(b_3 + b_1^2) + \frac{m(m-1)(m-2)(m-3)}{3!}{b_0}^{m-4}(b_4 + 3b_1 b_2) + \sum_{j\geq 5}^{\infty} b_0^{m-j} H_j. \tag{3}$$

and $H_j$with $(5 \leq j \leq m)$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, \dots, b_m$. (see [1]).

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The $q-$ calculus has attracted the attention of researchers due to its several applications in different branches of mathematics, especially in geometric function theory. Jackson ([10,11]) initiated and developed the application of $q-$ calculus. Chakrabarti and Jagannathan defined Jackson $(p,q)-$derivative as a generalization of q-derivative (see [8]). Al-Hawary et al. [5] introduced a new differential operator defined by the Jackson's $(p,q)-$derivative. Some applications of (p,q)- differential operators are studied by Altinkaya and Yalçın [6] and Araci et al. [7] .

For the expedience, we present some definitions and concepts of $(p,q)-$calculus that were used in this article by assuming p and q are fixed numbers such that $0 < p < q \leq 1$.

$$\partial_{p,q}h(z) = \begin{cases} \frac{h(pz)-h(qz)}{(p-q)z} & ,z \neq 0 \\ \partial_{p,q}h(0) = h'(0) & ,z = 0 \end{cases}. \tag{4}$$

provided $h'(0)$ exists, where the symbol, $[m]_{p,q}$denotes twin-basic number given by

$$[m]_{p,q} = \frac{p^m-q^m}{p-q},\ [0]_{p,q} = 0,\ [1]_{p,q} = 1. \tag{5}$$

Note that: For $0 < q < 1$ and $z \neq 0$, we have

- $\partial_{1,q}h(z) = \partial_q h(z) = \frac{h(qz)-h(z)}{qz-z}$, for more details, see [10]
- $[m]_{1,q} = [m]_q = \frac{1-q^m}{1-q} = \sum_{i=0}^{m-1} q^i$.

It's clear that for function h of the form (1), we have

$$\partial_{p,q}h(z) = 1 + \sum_{m=1}^{\infty} \frac{-[m]_{p,q}}{(pq)^m}\frac{a_m}{z^{m+1}}.$$

For $0 \leq \nu < 1$, $\xi \geq 1$, and $h \in \Sigma$, we define new subclass of meromorphic bi-univalent functions, denoted by $B\Sigma(\nu,\xi;p,q)$as:

Definition 1.1: A function h given by (11) is said to be in the class $B\Sigma(\nu,\xi;p,q)$if the following conditions hold true

$$Re\left\{(1-\xi)\frac{h(z)}{z} + \xi\partial_{p,q}h(z)\right\} > \nu, (z \in \Omega). \tag{6}$$

and

$$Re\left\{(1-\xi)\frac{\varphi(\omega)}{\omega} + \xi\partial_{p,q}\varphi(\omega)\right\} > \nu, (\omega \in \Omega). \tag{7}$$

where $0 \leq \nu < 1$, $\xi \geq 1$, and $\varphi = h^{-1}$.

We note from Definition 1.1 that

$$\lim_{p\to1^-} B\Sigma(\nu,\xi;p,q) = \left\{h: h \in \Sigma \text{ and } \begin{Bmatrix} Re[(1-\xi)\frac{h(z)}{z} + \xi\partial_q h(z)] > \nu \\ Re[(1-\xi)\frac{\varphi(\omega)}{\omega} + \xi\partial_q\varphi(\omega)] > \nu \end{Bmatrix}\right\} = B\Sigma(\nu,\xi;q)$$

Furthermore

$$\lim_{q\to1^-} B\Sigma(\nu,\xi;q) = \left\{h: h \in \Sigma \text{ and } \begin{Bmatrix} Re[(1-\xi)\frac{h(z)}{z} + \xi h'(z)] > \nu \\ Re[(1-\xi)\frac{\varphi(\omega)}{\omega} + \xi\varphi'(\omega)] > \nu \end{Bmatrix}\right\} = B\Sigma(\nu,\xi)$$

where the class $B\Sigma(\nu,\xi)$ is defined and studied by Hamidi [3].

In this paper, we obtain the bounds for the general coefficient $|a_m|$ of the class of meromorphic bi-univalent functions $B\Sigma(\nu,\xi;p,q)$. We also determine bounds for $|a_1|$,$|a_2|$, $|a_3|$ and for the combination $|a_2 + a_0a_1|$using Faber polynomial expansions.

2. PRELIMINARIES

In the following Theorem, we introduced an upper bounds for $|a_m|$for the class $B\Sigma(\nu,\xi;p,q)$.

Theorem 2.1: *Let* h *as in* (1). *For* $\xi \geq 1, 0 \leq \nu < 1$*and if*$h \in B\Sigma(\nu,\xi;p,q)$, *and*$a_k = 0$, $k = 0,1,\ldots,m-1$,*then*

$$|a_m| \leq \frac{2p^m q^m(1-\nu)}{(\xi-1)p^m q^m + \xi[m]_{p,q}}. \tag{8}$$

Proof. Let $h \in B\Sigma(\nu,\xi;p,q)$as in (1) then we have

$$(1-\xi)\frac{h(z)}{z} + \xi\partial_{p,q}h(z) = 1 + \sum_{m=0}^{\infty}\left(1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)\right)\frac{a_m}{z^{m+1}}. \tag{9}$$

and for $\varphi = h^{-1}$, we have

$$(1-\xi)\frac{\varphi(\omega)}{\omega} + \xi\partial_{p,q}\varphi(\omega) = 1 + \sum_{m=0}^{\infty}\left(1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)\right)\frac{b_m}{\omega^{m+1}} = 1-(1-\xi)\frac{b_0}{\omega} - \sum_{m=1}^{\infty}\left(1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)\right)\frac{1}{m}K_{m+1}^{m}(b_0,b_1,\ldots,b_m)\frac{1}{\omega^{m+1}}. \tag{10}$$

On the other hand, since $h \in B\Sigma(\nu,\xi;p,q)$, according to condition (6)implies that there exists a positive real part function $\sigma(z) = 1 + \sum_{m=1}^{\infty} c_m z^{-m} \in \Sigma$. So that,

$$\begin{aligned}(1-\xi)\frac{h(z)}{z} + \xi\partial_{p,q}h(z) &= \nu + (1-\nu)\sigma(z)\\ &= \nu + (1-\nu)\sum_{m=1}^{\infty}K_m^1(c_1,c_2,\ldots,c_{m+1})\frac{1}{z^m}.\end{aligned} \tag{11}$$

Similarly, for the inverse function$\varphi = h^{-1}$and according to condition(7), there exist a positive real part function$\chi(\omega) = 1 + \sum_{m=1}^{\infty} d_m\omega^{-m} \in \Sigma$. so that:

$$\begin{aligned}(1-\xi)\frac{\varphi(\omega)}{\omega} + \xi\partial_{p,q}\varphi(\omega) &= \nu + (1-\nu)\chi(\omega)\\ &= \nu + (1-\nu)\sum_{m=1}^{\infty}K_m^1(d_1,d_2,\ldots,d_{m+1})\frac{1}{z^m}.\end{aligned} \tag{12}$$

Comparing the corresponding coefficients of (9) and (11) yields to

$$\left(1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)\right)a_m = (1-\nu)\sum_{m=1}^{\infty}K_m^1(c_1,c_2,\ldots,c_{m+1}).$$

and similarly from (10) and (12) note that *for* $a_k = 0;\ 0 \leq k \leq m-1$with$a_m = -b_m$, we obtain:

$$\begin{cases}(1-\xi)a_0 = -(1-\nu)d_1\\ \dfrac{1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)}{m}K_{m+1}^{m}(a_0,a_1,\ldots,a_m) = -(1-\nu)K_m^1(d_1,d_2,\ldots,d_{m+1})\end{cases}. \tag{13}$$

and so

$$\begin{cases}\left[1-\xi\left(1+\dfrac{[m]_{p,q}}{p^m q^m}\right)\right]a_m = (1-\nu)c_{m+1}\\ \left[1-\xi\left(1+\dfrac{[m]_{p,q}}{p^m q^m}\right)\right]b_m = -(1-\nu)d_{m+1}.\end{cases} \tag{14}$$

By taking the absolute values of each above two equations and applying the CaratheodoryLemma(e.g., [2,9]),$|c_m| < 2$ and $|d_m| < 2$ for$m = 1,2,3,\ldots$,we get

$$|a_m| = \frac{(1-\nu)|c_{m+1}|}{\left|1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)\right|} = \frac{(1-\nu)|d_{m+1}|}{\left|1-\xi\left(1+\frac{[m]_{p,q}}{p^m q^m}\right)\right|} \leq \frac{2p^m q^m(1-\nu)}{(\xi-1)p^m q^m + \xi[m]_{p,q}}.$$

Corollary 2.1 *Leth as in* (1). *For* $\xi \geq 1, 0 \leq \nu < 1$ *and if* $h \in B\Sigma(\nu, \xi; q)$, *and* $a_k = 0$, $k = 0,1,\ldots,m-1$, *then*

$$|a_m| \leq \frac{2q^m(1-\nu)}{(\xi-1)q^m + \xi[m]_q}, m \geq 1.$$

Corollary 2.2 [3]*Let h as in (1). For* $\xi \geq 1, 0 \leq \nu < 1$ *and if* $h \in B\Sigma(\nu, \xi)$, *and* $a_k = 0$, $k = 0,1,\ldots,m-1$ *then*

$$|a_m| \leq \frac{2(1-\nu)}{\xi(n+1)-1}, \quad m \geq 1.$$

By relaxing the coefficient restrictions imposed on Theorem 2.1 we obtain estimates for early coefficient of functions $h \in B\Sigma(\nu, \xi; p, q)$, and the combination $|a_2 + a_0 a_1|$.

Theorem 2.2 *For* $\xi \geq 1$, $0 \leq \nu < 1$, *and h of the form (1) be in the class* $B\Sigma(\nu, \xi; p, q)$, *then we have the following consequence.*

$$|a_0| \leq \frac{2(1-\nu)}{\xi - 1},$$
$$|a_1| \leq \frac{2pq(1-\nu)}{\xi(pq+1) - pq},$$
$$|a_2| \leq \frac{2p^2q^2(1-\nu)}{\xi(p^2+pq+q^2) - p^2q^2},$$
$$|a_2 + a_1a_0| \leq \frac{2p^2q^2(1-\nu)}{\xi(p^2+pq+q^2) - p^2q^2}.$$

Proof. Let $h \in B\Sigma(\nu, \xi; p, q)$ as in (1), and compare the Eqs. (9) and (11) for $m = 0,1$ and $m = 2$, we get

$$(1-\xi)a_0 = (1-\nu)c_1 \tag{15}$$
$$\left(1 - \xi\left(1 + \frac{1}{pq}\right)\right)a_1 = (1-\nu)c_2 \tag{16}$$
$$\left(1 - \xi\left(\frac{p^2+pq+q^2}{pq}\right)\right)a_2 = (1-\nu)c_3 \tag{17}$$

and from Equations (10) and (12), for $m = 2$, we have

$$-\left(p^2q^2 - \xi(p^2+pq+q^2)\right)(a_2 + a_0a_1) = p^2q^2(1-\nu)d_3 \tag{18}$$

By solving equations (15), (16), (17) and (18) for a_0, a_1, a_2 and $a_2 + a_0a_1$, respectively, and taking the absolute value then applying Caratheodory Lemma, we will get

$$|a_0| = \frac{(1-\nu)|c_1|}{|1-\xi|} \leq \frac{2(1-\nu)}{\xi - 1},$$
$$|a_1| = \frac{pq(1-\nu)|c_2|}{|pq - \xi(pq+1)|} \leq \frac{2pq(1-\nu)}{\xi(pq+1) - pq},$$
$$|a_2| = \frac{p^2q^2(1-\nu)|c_3|}{|p^2q^2 - \xi(p^2+pq+q^2)|} \leq \frac{2p^2q^2(1-\nu)}{\xi(p^2+pq+q^2) - p^2q^2},$$

and

$$|a_2 + a_1a_0| \leq \frac{2p^2q^2(1-\nu)}{\xi(p^2+pq+q^2) - p^2q^2}.$$

By letting $p \to 1^-$in Theorem 2.2, we obtain the following consequence.

Corollary 2.3 *Let h of the form (1) be in the class* $B\Sigma(\nu,\xi;q)$*, and for* $\xi \geq 1$ *and* $0 \leq \nu < 1$ *then*

1) $|a_0| \leq \frac{2(1-\nu)}{\xi-1}$,
2) $|a_1| \leq \frac{2q(1-\nu)}{\xi(q+1)-q}$,
3) $|a_2| \leq \frac{2q^2(1-\nu)}{\xi(1+q+q^2)-q^2}$,
4) $|a_2 + a_1 a_0| \leq \frac{2q^2(1-\nu)}{\xi(1+q+q^2)-q^2}$.

For$q \to 1^-$in Corollary 2.3, we obtain the following consequence.

Corollary 2.4 *[3] Let h of the form (1) be in the class* $B\Sigma(\nu,\xi)$*,and For* $\xi \geq 1$ *and* $0 \leq \nu < 1$ *then*

1) $|a_0| \leq \frac{2(1-\nu)}{\xi-1}$,
2) $|a_1| \leq \frac{2(1-\nu)}{2\xi-q}$,
3) $|a_2| \leq \frac{2(1-\nu)}{3\xi-1}$,
4) $|a_2 + a_1 a_0| \leq \frac{2(1-\nu)}{3\xi-1}$.

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ON THE BEHAVIOR OFSOLUTIONS AND LIMITOFTWODIMENSIONALDECOUPLED SYSTEMS OF DIFFERENCE EQUATIONS

$x_{n+1} = \frac{x_n}{x_{n-1}+r}, y_{n+1} = \frac{x_n y_n}{x_{n-1}y_{n-1}+r}$ and $x_{n+1} = \frac{x_n}{x_n+1}$, $y_{n+1} = \frac{x_{n-1}y_n}{x_{n-1}y_n+1}$

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ABSTRACT

In this paper we study systems of difference equations numerically and theoretically. These systems were considered by many researchers. We will focus on the general form and the limits. We consider different orders of the difference systems. We use in certain cases the computer to verify the limit properties.

Keywords: difference equations; limit; Gamma function

1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena because measurements of time evolving variables are discrete and as such, these equations are in their own right important mathematical models. More importunately, difference equations also appear in the study of discrimination methods for difference equations. Several results in the theory of difference equation have been obtained as more or less natural discrete analogues of corresponding results of difference equation. Recently many researchers worked in the topic of the behavior of the solution of difference equations. In the literature we can find the works of them such as Kurbanli, El- Metwally, Amleh, Elabbasy and Elsayed.

In [7] El-Metwally, Elabbasy and Elsayed studied the following difference equation

$$x_{n+1} = max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}.$$

They found the general form of the solution in some cases, also They proved that every positive solution of this equation is bounded. In [3] Elsayed computed the general form of the solutions of difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}\,x_{n-3}x_{n-5}}.$$

Further, he proved that every positive solution of this equation is bounded and

$$\lim_{n\to\infty} x_n = 0$$

In [1]Abuhayal considered the following system of difference equations:

$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}+r}, \quad y_{n+1} = \frac{x_{n-1}y_n}{x_{n-1}y_n+r}$$

Abuhayal calculated the solution for the system with the following initial values:

$$x_0 = a, x_{-1} = b, y_0 = c$$

In this solution we distinguish between odd and even terms. In [8]Yaqoub considered the following system of difference equations:

$$x_{n+1} = \frac{x_{n-1}}{x_n+r}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}+r}$$

Yaqoub proved the following result: Let $r = 1$ and a, b, c, d be real numbers. The solution for the system with the following initial values:

$$x_{-1} = a, x_0 = 0, y_{-1} = b, y_0 = d$$

is

$$x_{2n} = 0 \ , y_{2n} = \frac{d}{ab + (n-1)ad + 1}, x_{2n+1} = a \quad , y_{2n+1} = \frac{d}{ab + nad + 1}$$

In [6] the following system of equations was studied by Ibrahim

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{x_{n-1}}{x_{n-1} + r} \\ \dfrac{y_{n-1}}{x_n y_{n-1} + r} \end{pmatrix}$$

where$r$ is a fixed real number. With the following initial condition

$$x_0 = b, x_{-1} = c, y_0 = a, y_{-1} = d \ .$$

In [6] Ibrahim proved the following result: Let *a* ,*b* ,*c* ,*d*, *r* be positive real numbers. Then, the general solution of the system is

$$x_{2k} = \frac{b}{G(b,k)}, x_{2k+1} = \frac{c}{G(c,k+1)},$$

$$y_{2k} = \frac{ac^k}{ac^k + a\sum_{i=2}^{k} c^{k-i+1} r^{i-1} \prod_{j=0}^{i-2} G(c,k-j) + r^k \prod_{j=0}^{k-1} G(c,k-j)},$$

$$y_{2k+1} = \frac{db^{k+1}}{db^{k+1} + db\sum_{i=2}^{k} b^{k-i+1} r^{i-1} \prod_{j=0}^{i-2} G(b,k-j) + r^k \prod_{j=0}^{k-1} G(b,k-j)},$$

where

$$G(c,0) = c + r, G(c,i) = c + rG(c,i-1).$$

In [4] Bany Khaled considered the system

$$x_{n+1} = \frac{x_{n-1}}{x_n + r}, y_{n+1} = \frac{x_{n-1} y_{n-1}}{x_{n-1} y_{n-1} + r}$$

with initial values

$$x_{-1} = a, x_0 = 0, y_{-1} = b,$$

Hence, according to definition we obtain

$$x_{2k} = 0, y_{2k} = 0.$$

Bany Khaled proved an estimate for the solution. Based on it she proved: If $a, b > 0$ and $r > 1$such that $a^2 < r$, then $\lim_{k\to\infty} x_{2k+1} = 0, \lim_{k\to\infty} y_{2k+1} = 0.$

2. MAINRESULTS

In this paper we consider thefollowing three systems

$$x_{n+1} = \frac{x_n}{x_{n-1} + r}, y_{n+1} = \frac{x_n y_n}{x_{n-1} y_{n-1} + r} \quad (1)$$

$$x_{n+1} = \frac{x_n}{x_n + 1} \ , \ y_{n+1} = \frac{x_{n-1} y_n}{x_{n-1} y_n + 1} \quad (2)$$

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} + r} \ , \ y_{n+1} = \frac{x_{n-1} y_{n-1}}{x_{n-1} y_{n-1} + r} \quad (3)$$

We define

$$W(p,f) = \sum_{k=0}^{f} \frac{1}{\Gamma(k+p)}, \quad R(b) = \Gamma(b) - \Gamma(b,1).$$

We verified the following result by Mathematica for *p*>0:

$$\sum_{j=0}^{f} \frac{1}{\Gamma(j+p)} = e\frac{(p-1)R(p-1)}{\Gamma(p)} - e\frac{(f+p)R(f+p)}{\Gamma(f+1+p)} \quad (4)$$

where*e* is the Euler number (approx. 2.718) and Γ(*a*,*x*) is the incomplete gamma function.

7.1. *The limit of system (1)*

We consider the system (1) just in case of positive initial values and *r*. We will study first the following equation since this equation is separated than the second one.

Lemma 2.1.*Suppose*$x_{-1}, r > 0, x_0 = a > 0$. *Then*$x_n < ar^{-n}$, *n*= 1,2, ...

Proof: We start with

$$x_1 = \frac{x_0}{x_{-1}+r} = \frac{a}{x_{-1}+r} < \frac{a}{r} = ar^{-1} \text{since} x_{-1} + r > r > 0.$$

We consider this relation as basis step. We continue by induction: Suppose that $x_k < ar^{-k}$ for some integer *k*. Then according to definition and that $x_{k-1} > 0$

$$x_{k+1} = \frac{x_k}{x_{k-1}+r} < \frac{x_k}{r} < \frac{ar^{-k}}{r} = \frac{a}{r^{k+1}} \square$$

After some calculations we prove

Theorem 2.2*Assume* $r, x_{-1}, y_{-1}, x_0, y_0 > 0$. *Then* $\lim_{r\to\infty} x_n = 0, \lim_{n\to\infty} y_n = 0$.

We consider a special case, namely $r = 0$.In this case it is easy to compute the general solution. If we take the initial values

$$x_{-1} = a\ , x_0 = c\ , y_{-1} = b\ , y_0 = d$$

Then we obtainfor $n = 1,2, \ldots$

$$x_{6n-2} = \frac{a}{c}, y_{6n-2} = \left(\frac{a}{c^2}\right)^{2n}\frac{cb}{d}, \quad x_{6n-1} = a, y_{6n-1} = \left(\frac{a^2}{c}\right)^{2n} d,$$

and for $n = 0,1,2, \ldots$

$$x_{6n} = c, y_{6n} = (ac)^{2n} d, x_{6n+1} = \frac{c}{a}, y_{6n+1} = \left(\frac{c^2}{a}\right)^{2n}\frac{cd}{ab},$$

$$x_{6n+2} = \frac{1}{a}, \ y_{6n+2} = \left(\frac{c}{a^2}\right)^{(2n+1)}\frac{1}{b}, x_{6n+3} = \frac{1}{c}, \qquad y_{6n+3} = \frac{1}{(ac)^{2n+1}d}.$$

We notice that we have a periodic solution, which consists of 6 elements. This is an essential change in the behavior of the sequence. It is an open problem, what will happen if *r* is negative.

7.2. *The general solution of system (2)*

We study now the system (2) with initial values

$$x_0 = a, x_{-1} = b, y_0 = c.$$

We find that in general

$$x_n = \frac{a}{na+1}, y_n = \frac{a^{n-1}bc}{P_n}, \text{for } n = 1,2, \ldots$$

$$P_{n+1} = a^n bc + \big((n-1)a + 1\big)P_n\ , \quad P_1 = bc + 1.$$

Hence

$$P_n = a^{n-1} bc \Gamma\left(n - 2 + \frac{1+a}{a}\right) * W(a, n-2) + Z,$$

where

$$Z = \frac{a^{n-2}\Gamma\left(n - 2 + \frac{1+a}{a}\right)}{\Gamma\left(\frac{1+a}{a}\right)} P_1.$$

We reachthe following result

Proposition 2.1 *The general solution of the system* (2) *is*

$$x_1=\frac{a}{a+1},y_1=\frac{bc}{bc+1}x_n=\frac{a}{na+1},$$

$$y_n=\frac{abce^{-1}\Gamma(1+a^{-1})\Gamma(n+a^{-1})}{(n+a^{-1}-1)(\Gamma(n+a^{-1}-1)R(a^{-1})-\Gamma(a^{-1})R(n+a^{-1}-1))+\Gamma(n+a^{-1}-1)}.$$

Proof. We concluded previously

$$y_n=\frac{\Gamma(1+a^{-1})}{\Gamma(n-1+a^{-1})}\frac{a^{n-1}bc}{a^{n-1}bc\Gamma(1+a^{-1})W(a,n-2)+a^{n-2}(bc+1)}$$

If we set $p=\frac{1+a}{a}=1+a^{-1}$in(4), then we obtainfor *n*=2, 3, …

$$W(a,n-2)=\frac{a^{-1}e(\Gamma(a^{-1})-\Gamma(a^{-1},1))}{\Gamma(1+a^{-1})}-\frac{e(n-1+a^{-1})(\Gamma(n-1+a^{-1})-\Gamma(n-1+a^{-1},1))}{\Gamma(n+a^{-1})}.$$

$$y_n=\frac{\Gamma(1+a^{-1})}{\Gamma(n-1+a^{-1})}\frac{abc}{abc\Gamma(1+a^{-1})W(a,n-2)+bc+1}=\frac{abce^{-1}\Gamma(1+a^{-1})\Gamma(n+a^{-1})}{H}$$

where

$H=\Gamma(n-1+a^{-1})+bc[\Gamma(n+a^{-1})R(a^{-1})+\Gamma(n-1+a^{-1})-(an+1-a)\Gamma(1+a^{-1})R(n+a^{-1}-1)]=\Gamma(n+a^{-1}-1)a^{-1}\big((an+1-a)R(a^{-1})+a\big)-(an+1-a)\Gamma(1+a^{-1})R(n+a^{-1}-1)=(an+1-a)\big(a^{-1}\Gamma(n+a^{-1}-1)R(a^{-1})-\Gamma(1+a^{-1})R(n+a^{-1}-1)\big)+\Gamma(n+a^{-1}-1)=$
$(n+a^{-1}-1)(\Gamma(n+a^{-1}-1)R(a^{-1})-\Gamma(a^{-1})R(n+a^{-1}-1))+\Gamma(n+a^{-1}-1)$

since

$\Gamma(n+a^{-1})R(a^{-1})+\Gamma(n-1+a^{-1})=\Gamma(n+a^{-1}-1)((n-1+a^{-1})R(a^{-1})+1)=$
$\Gamma(n+a^{-1}-1)a^{-1}\big((an+1-a)R(a^{-1})+a\big)$. □

Corollary 2.2 *If a> 0, then the solution of the system* (2) *tends to*

$$\frac{abc\Gamma(1+a^{-1})}{eR(a^{-1})}.$$

Proof. We know

$$y_n=\frac{abce^{-1}\Gamma(1+a^{-1})}{(n+a^{-1}-1)(\frac{\Gamma(n+a^{-1}-1)}{\Gamma(n+a^{-1})}R(a^{-1})-\frac{\Gamma(a^{-1})}{\Gamma(n+a^{-1})}R(n+a^{-1}-1))+\frac{\Gamma(n+a^{-1}-1)}{\Gamma(n+a^{-1})}}=\frac{abce^{-1}\Gamma(1+a^{-1})}{R(a^{-1})-\frac{\Gamma(a^{-1})}{\Gamma(n+a^{-1}-1)}R(n+a^{-1}-1)+\frac{1}{n+a^{-1}-1}}$$

Since

$$R(b)=\Gamma(b)-\Gamma(b,1)=\int_0^1 e^{-u}u^{b-1}du,$$

$$|R(n+a^{-1}-1)|\le\int_0^1 u^{n+a^{-1}-2}du=\frac{1}{n+a^{-1}-1}\to 0\text{as}n\to\infty$$

So, we are done□

7.3. *The general solution of system (3) in case r* = 1

We consider now the system (3) withthe following initial values

$$x_{-1} = a, x_0 = c, y_{-1} = b, y_0 = d$$

Proposition 2.3 *If a> 0, thenthe general solution of the system* (3) *is*

$$x_{2k} = \frac{c}{ck+}, \quad x_{2k+1} = \frac{a}{a(k+1)+1} \quad ,$$

$$y_{2k} = \frac{\Gamma(\frac{1}{c})}{\Gamma(\frac{1}{c}) + \Gamma(\frac{1}{c})\Gamma(k+\frac{1}{c})W(\frac{1}{c}+2, k-3) + \Gamma(k+\frac{1}{c})(c+\frac{1}{d})},$$

$$y_{2k+1} = \frac{\Gamma(\frac{1}{a})}{\Gamma(\frac{1}{a}) + \Gamma(\frac{1}{a})\Gamma(k+\frac{1+a}{a})W(\frac{1}{a}+2, k-2) + \Gamma(k+\frac{1+a}{a})(a+\frac{1}{b})} \text{ for } k = 3,4,\ldots$$

Proof. According to definition

$$x_1 = \frac{x_{-1}}{x_{-1}+r} = \frac{a}{a+r} = \frac{a}{G(1)}, y_1 = \frac{x_{-1}y_{-1}}{x_{-1}y_{-1}+r} = \frac{ab}{ab+r} = \frac{ab}{H(1)}$$

where we denote by $G(n)$ (res. $H(n)$) the denominator of x_n (res. y_n). Since the variables x_n and y_n are separated in the even and the odd cases we are going to consider just one case. Now, we obtain

$$x_2 = \frac{x_0}{x_0+r} = \frac{c}{c+r} = \frac{c}{G(2)}, \; x_3 = \frac{x_1}{x_1+r} = \frac{\frac{a}{G(1)}}{\frac{a}{G(1)}+r} = \frac{a}{a+rG(1)} = \frac{a}{G(3)}, \ldots,$$

$$y_7 = \frac{x_5y_5}{x_5y_5+r} = \frac{\frac{a}{G(5)} * \frac{a^3b}{H(5)}}{\frac{a}{G(5)} * \frac{a^3b}{H(5)}+r} = \frac{a^4b}{a^4b+rG(5)H(5)} = \frac{a^4b}{H(7)}$$

In general we denote by

$$G_j(a) = (r^{j-1}+\ldots.+r+1)a + r^j = aj+1$$

since $r=1$. We conclude that

$$x_{2k} = \frac{c}{G_k(c)}, x_{2k+1} = \frac{a}{G_{k+1}(a)}, y_{2k+1} = \frac{a^{k+1}b}{H(2k+1)}, y_{2k} = \frac{c^kd}{H(2k)},$$

$$H(1) = ab+r, \; H(3) = a^2b + rG(1)H(1) = a^2b + rG_1(a)(ab+r),$$

$$H(5) = a^3b + rG(3)H(3) = a^3b + rG_2(a)(a^2b + rG_1(a)(ab+r))$$

$$= a^3b + a^2rbG_2(a) + r^2G_2(a)G_1(a)(ab+r),$$

$$H(7) = a^4b + rG(5)H(5) = a^4b + rG_3(a)(a^3b + a^2rbG_2(a) + r^2G_2(a)G_1(a)(ab+r))$$

$$= a^4b + a^3brG_3(a) + a^2r^2bG_2(a)G_3(a) + r^3G_2(a)G_1(a)G_3(a)(ab+r).$$

We use the notation

$$B_n(a) = \prod_{j=1}^{n} G_j(a)$$

We rewrite

$$H \quad * \quad + \quad = a^4b + a^3br\frac{B_3 \; a}{B_2 \; a} + a^2r^2b\frac{B_3 \; a}{B_1 \; a} \ldots$$ [illegible]

Thus the general form for $k = 3,4,5,\ldots$

$$H(2k+1)=a^{k+1}b+\sum_{i=1}^{k-1}a^{k+1-i}br^{i}\frac{B_k(a)}{B_{k-i}(a)}+r^kB_k(ab+r),$$

$$\sum_{i=1}^{k-1}a^{k+1-i}br^{i}\frac{B_k(a)}{B_{k-i}(a)}=a^{k+1}bB_k(a)\sum_{i=1}^{k-1}\frac{a^{-i}r^{i}}{B_{k-i}(a)}$$

Since $r=1$

$$B_n(a)=\prod_{j=1}^{n}G_j(a)=\prod_{j=1}^{n}(aj+1)$$

But

$$\prod_{l=0}^{n}(p+ql)=q^{n+1}\Gamma(n+\tfrac{q+p}{q})\Gamma^{-1}(\tfrac{p}{q})$$

Hence,

$$B_n(a)=a^{n+1}\Gamma(n+\tfrac{1+a}{a})\Gamma^{-1}(\tfrac{1}{a}),$$

$$a^{k+1}bB_k(a)\sum_{i=1}^{k-1}\frac{a^{-i}r^{i}}{B_{k-i}(a)}=a^{k+1}ba^{k+1}\frac{\Gamma(k+\frac{1+a}{a})}{\Gamma(\frac{1}{a})}\sum_{i=1}^{k-1}\frac{a^{-i}\Gamma(\frac{1}{a})}{a^{k-i+1}\Gamma(k-i+\frac{1+a}{a})}=$$

$$ba^{k+1}\,\Gamma(k+\frac{1+a}{a})\sum_{i=1}^{k-1}\frac{1}{\Gamma(k-i+\frac{1+a}{a})}=ba^{k+1}\,\Gamma(k+\frac{1+a}{a})\sum_{i=2}^{k}\frac{1}{\Gamma(i+\frac{1}{a})},$$

$$H(2k+1)=a^{k+1}b+ba^{k+1}\,\Gamma(k+\frac{1+a}{a})W(\frac{1}{a}+2,k-2)+r^ka^{k+1}\Gamma(k+\tfrac{1+a}{a})\Gamma(\tfrac{1}{a})^{-1}(a+\frac{1}{b})$$

$$H(2k+1)=a^{k+1}b\left(1+\Gamma(k+\frac{1+a}{a})\right)W(\frac{1}{a}+2,k-2)+a^{k+1}\Gamma(k+\tfrac{1+a}{a})\Gamma(\tfrac{1}{a})^{-1}(a+\frac{1}{b}),$$

$$y_{2k+1}=\frac{b\Gamma(a^{-1})}{\Gamma(a^{-1})(1+\Gamma(k+1+a^{-1}))W(2+a^{-1},k-2)+\Gamma(k+1+a^{-1})(a+b^{-1})}$$

Similarly we can prove the other case.□

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$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}-1},\ \text{and } z_{n+1}=\frac{z_{n-1}}{y_nz_{n-1}-1}$$

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ON THE WEIGHTED MIXED ALMOST UNBIASED LIU TYPE ESTIMATOR

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ABSTRACT

This paper deals with a new version of weighted mixed estimator based on prior information in stochastic linear restricted model for the unknown vector parameter when stochastic linear restrictions on the parameters hold. The performance of the proposed estimator as a generalization of the weighted mixed estimator (WME), the almost unbiased Liu estimator (AULE) and the least squares estimator (LSE) has been given in terms of the mean squares error matrix. Finally, numerical example from literature and simulation study have been given to illustrate the results.

Keywords: Mixed model; Stochastic linear restrictions

1. INTRODUCTION

We consider the standard multiple linear regression model

$$Y = X\beta + \epsilon, \qquad (1)$$

Where Y is an n x 1 vector of observations on the response (or dependent) variable, X is an n x p model matrix of observations on p non-stochastic explanatory variables, β is a p x 1 vector of unknown parameters associated with the p explanatory variables and ϵ is an n x 1 vector of residuals with expectation E(ϵ) = 0 and dispersion matrix $Var(\epsilon) = \sigma^2 I_n$.
It is well known that, the least squares is the best method for fitting model (1) . The least squares estimator (LSE) is define as:

$$\hat{\beta} = S^{-1}X'Y, \qquad (2)$$

Where $S = X'X$. the LSE in (2) is unbiased and has minimum variance among all linear unbiased estimators when it satisfy it's conditions and one of these conditions is no high correlation between the independent variables. However, This is not the case many when the multicollinearity is present where there are many results have proved that the LSE is no longer a good estimator.
To reduce the effect of multicollinearity, several techniques have been proposed. One of them is biased estimation technique that used as an alternative to LSE to obtain some reduction in the variance with some cost in the bias. Hoerl and Kennard (1970) proposed the ridge estimator (RE) as

$$\hat{\beta}_k = [S + kI_p]^{-1}X'Y = [I_p + kS^{-1}]^{-1}\hat{\beta},$$

Where k > 0. Liu (1993) proposed Liu estimator (LE) as

$$\hat{\beta}_d = (S + I)^{-1}(S + dI)\hat{\beta},$$

Where 0 < d <1.
Since $X'X$ is symmetric, there exists a pxp orthogonal matrix P such that $P'X'XP = \Lambda$, Λ is a pxp diagonal matrix where diagonal elements $\lambda_1 \ldots \lambda_p$ are the eigenvalues of $X'X$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_p$. So, model (1) can be written in the canonical form as :

$$Y = Z\alpha + \epsilon, \qquad (3)$$

Where Z= XP and $\alpha = P'\beta$. Therefore, The LSE and LE are respectively

$$\hat{\alpha} = \Lambda^{-1}Z'Y \qquad (4)$$

And

$$\hat{\alpha}_d = (\Lambda + I)^{-1}(\Lambda + dI)\hat{\alpha}. \qquad (5)$$

In order to reduce the cost of the bias in biased estimators with small change in the variance, Singh et al. (1986) introduced the almost unbiased ridge estimator (AURE) as:

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$$\hat{\alpha}_{AURE}(k) = [\,I - k^2(\Lambda + kI)^{-2}]\hat{\alpha} \qquad (6)$$

Also, Akdeniz and Kaciranlar (1995) proposed the almost unbiased generalized Liu estimator (AULE)

$$\hat{\alpha}_{AULE}(d) = [I - (\Lambda + I)^{-2}(1-d)^2]\hat{\alpha}. \qquad (7)$$

In addition to model (1), we suppose that, there are some prior information about β in the form of a set of independent stochastic linear restrictions

$$r = R\beta + \epsilon^*, \qquad (8)$$

Where R is an q x p non zero matrix with rank (R) = q < p, r is an q x 1 known vector which is interpreted as a random variable with E(r) =Rβ and ϵ^* is an q x 1 vector of disturbances with zero mean and variance-covariance matrix $\sigma^2 V$, V is known and positive definite .

Also (8) can be transformed into the canonical form $T\alpha = r$ where T= RP . It is clear that, the stochastic restrictions in (8) do not hold exactly but will hold at the mean . Further, it is also assumed that ϵ^* is stochastically independent of ϵ . By unifying the sample and prior information in a common model (see Rao et al. , 2008)

$$\binom{Y}{r} = \binom{Z}{T}\alpha + \binom{\epsilon}{\epsilon^*}, \qquad (9)$$

Where $E(\epsilon\epsilon^{*\prime}) = 0$ and $\binom{\epsilon}{\epsilon^*}(\,\epsilon\;\epsilon^*) = \sigma^2\begin{pmatrix} I & 0 \\ 0 & V\end{pmatrix}$, we can use the least squares method for model (9) to get the mixed estimator (ME) which is introduced by Theill and Goldberger(1961). The ME is defined as follows :

$$\hat{\alpha}_{ME} = (\Lambda + R'V^{-1}R)^{-1}(Z'Y + R'V^{-1}r)\,. \qquad (10)$$

Since we assumed the stochastic restrictions are held, i.e. E(r) – Tα = 0, the mixed estimator is unbiased .

In case the prior information and sample information are not equally important in model (1) with stochastic linear restrictions in (8) , Schffrin and Toutenburg (1990) introduced the weighted mixed estimator (WME) as follows:

$$\hat{\alpha}_w = (\Lambda + wR'V^{-1}R)^{-1}(Z'Y + wR'V^{-1}r)\,, \qquad (11)$$

where $0 \le w \le 1$ is a scalar weight.

Chaolin Liu et al.(2013) proposed the weighted mixed almost unbiased ridge estimator as follows:

$$\hat{\alpha}_{MAURE}(k) = \hat{\alpha}_{AURE}(k) + \Lambda^{-1}R'(V + R\Lambda^{-1}R')(r - R\,\hat{\alpha}_{AURE}(k))$$
$$= (\Lambda + R'V^{-1}R)^{-1}(GZ'Y + R'V^{-1}r),$$

where $\hat{\alpha}_{AURE}(k) = [\,I - k^2(\Lambda + kI)^{-2}]\hat{\alpha}$ and $G = I - k^2(\Lambda + I)^{-2}$.

In this paper, we introduce a new type of weighted mixed estimator as a generalization of some other estimators. The proposed estimator and its properties is given in Section 2 . In section 3 the performance of the new estimator compared with other estimators with respect to the mean squares error matrix as a criteria are given .

2. THE NEW ESTIMATOR AND ITS PROPERTIES

In the first, let us give some bases information that can help us to understand the proposed work in this paper.

Lemma 1 : (See Rao et al. 2008) Let A: pxp, B:pxn, C:nxn and D:nxp. If all the inverses exist, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

Lemma 2 : (See Farebrother 1979) Lel A be a p.d. matrix , c be an non zero vector and θ be a positive scaler . Then $\theta A - cc'$ is p.d. if and only if $c'A^{-1}c < \theta$

Lemma 3: (See Rao et al. 2008) Let $\hat{\beta}_j = A_jY$, j=1,2 be two linear estimators of β. Suppose that $D = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2)$ is p.d. then $\Delta = MSE(\hat{\beta}_1) - MSE(\hat{\beta}_2)$ is n.n.d. if and only if $b_2'(D + b_1b_1')^{-1}b_2 \le 1$, where b_j denotes the bias vector of $\hat{\beta}_j$.

Lemma 4 : (Hu Yang et al., 2009)

Suppose A is a real symmetric matrix, P is a matrix then $A \ge 0 \Leftrightarrow \forall P, P'AP \ge 0 \Leftrightarrow$ each eigenvalue of A is non negative .

Using lemma 1, the WME estimator can be rewritten as follows :

$$\hat{\alpha}_w = \hat{\alpha} + w\Lambda^{-1}R'(V + wR\Lambda^{-1}R)^{-1}(r - R\hat{\alpha}). \qquad (12)$$

Now, if we replace $\hat{\alpha}$ with $\hat{\alpha}_{AULE}(d)$, we get the new proposed estimator as follows:

$$\hat{\alpha}_{WMAULE}(d) = \hat{\alpha}_{AULE}(d) + w\Lambda^{-1} R'(V + wR\Lambda^{-1} R')^{-1}(r - R\ \hat{\alpha}_{AULE}(d))$$
$$= (\Lambda + wR'V^{-1}R)^{-1}(JZ'Y + wR'V^{-1}r) \quad (13)$$

Where $J = I - (1-d)^2(\Lambda + I)^{-2}$.

We are calling as the weighted mixed almost unbiased Liu estimator (WMAULE).

Remark: As we mention in the first, the reason for considering the AULE is to reduce the bias of LE, at the same time there is a gain in the variance. Therefore, the hope these advantages will inherit to WMAULE .

The WMAULE is general estimator that includes the LSE, the ME and the AULE estimators:

$$\hat{\alpha}_{WMAULE}(1) = \hat{\alpha}_{w}$$

If R=0, then

$$\hat{\alpha}_{WMAULE}(d) = \hat{\alpha}_{AULE}(d);$$

And when w=1;

$$\hat{\alpha}_{WMAULE}(1) = \hat{\alpha}$$

The properties of the proposed estimator can be easily computed. Therefore, the expected value, the variance and the bias of the WMAULE are given as follows:

$$E(\hat{\alpha}_{WMAULE}(d)) = -(1-d)^2A(\Lambda + I)^{-2}\Lambda\alpha + \alpha$$
$$Var(\hat{\alpha}_{WMAULE}(d)) = \sigma^2\ A(J\Lambda J' + w^2R'V^{-1}R)A'$$
$$Bias(\hat{\alpha}_{WMAULE}(d)) = -(1-d)^2A(\Lambda + I)^{-2}\Lambda\alpha$$
$$= c_1$$

Where $A = (\Lambda + wR'V^{-1}R)^{-1}$. The bias and the variance of an estimator β^* is measured simultaneously by the mean squares error matrix (MSE)

$MSE(\beta^*) = Var(\beta^*) + Bias\ (\beta^*)(Bias(\beta^*))'$.

For this purpose ,

$$MSE(\hat{\alpha}_{w}) = \sigma^2A(\Lambda + w^2R'V^{-1}R)A'. \quad (14)$$
$$MSE(\hat{\alpha}_{WMAULE}(d)) = \sigma^2A(J\Lambda J' + w^2R'V^{-1}R)A' + c_1c'_1 (15)$$

3. *SUPERIORITY OF THE NEW ESTIMATORS*

Let $\beta_i^* = A_iY$, i=1,2 be any two estimators. We know that

$$MSE(\beta_1^*) - MSE(\beta_2^*) = Var(\beta_1^*) - Var(\beta_2^*) + B_1B_1' - B_2B_2'$$
$$= \sigma^2D + B_1B_1' - B_2B_2',$$

Where $D = A_1A_1' - A_2A_2'$. If we want to know whether $\Delta = MSE(\beta_1^*) - MSE(\beta_2^*)$ is a positive definite (p.d.) or not , we may confine ourselves to the following fact :

Since $B_1B_1'$is a non negative definite (n.n.d) matrix and D is a p.d. This implies that $\sigma^2D + B_1B_1'$ is a p.d. (see Rao et.al.2008). Thus, Δ reduce to the matrix type $\theta A - cc'$. Therefore, Δ is p.d. when A is p.d.

Let us consider the difference among the estimators:

$$\Delta_1 = MSE(\hat{\alpha}_{ME}) - MSE(\hat{\alpha}_{MAULE}(d)) = \sigma^2D_2 - c_1c_1',$$

$$\Delta_2 = MSE(\hat{\alpha}_{MAURE}(k)) - MSE(\hat{\alpha}_{MAULE}(d)) = \sigma^2D_2 + b_1b_1' - c_1c_1'$$

Where

$D_1 = A(\Lambda + w^2R'V^{-1}R)A' - A(J\Lambda J' + w^2R'V^{-1}R)A',$

$D_2 = A(G\Lambda G' + w^2R'V^{-1}R)A' - A(J\Lambda J' + w^2R'V^{-1}R)A'.$

3.1 *Superiority of the mixed almost unbiased Liu estimator*

We are searching now for the condition that makes the proposed estimator is better than ME . For this reason , we need to check when $D_1$is p.d.

D1 can be written as following :

$$.D_1 = A\Lambda(I - JJ')A'$$

But $I - JJ' = I - [I - (\Lambda + I)^{-2}(I - dI)^2]^2$and each element of it is $1 - [1 - \frac{(1-d)^2}{(\lambda_i+1)^2}]^2$. When $0 < d < 1$, it is clear that $1 - [1 - \frac{(1-d)^2}{(\lambda_i+1)^2}]^2 < 1$ and that means D1 is p.d. Therefore we have the following theorem:

Theorem 1
The proposed MAULE is superior to the ME in the MSE sense, namely, Δ_2 if and only if $c_1' D_2^{-1} c_1 \leq \sigma^2$.

Let us rewrite D2 as follows:

$$D_2 = A\Lambda(GG' - JJ')A'$$

But ,

$$GG' - JJ' = [I - k^2(\Lambda + kI)^{-2}]^2 - [I - (\Lambda + I)^{-2}(I - dI)^2]^2 \qquad (17)$$

For any element i=1,…..,p, the elements of (17) will be in the form $(1 - \frac{K^2}{(\lambda_i+K)^2})^2 - (1 - \frac{(1-d)^2}{(\lambda_i+1)^2})^2$. Therefore the condition that makes D2 p.d. is reduced to the condition that makes $(1 - \frac{K^2}{(\lambda_i+K)^2})^2 - (1 - \frac{(1-d)^2}{(\lambda_i+1)^2})^2 > 0$. Let d be fixed for the moments, the condition $(1 - \frac{K^2}{(\lambda_i+K)^2})^2 - (1 - \frac{(1-d)^2}{(\lambda_i+1)^2})^2 > 0$ will reduce to condition $(\lambda_i + k)(1 - d) - k(\lambda_i + 1) > 0$ and this will satisfy when $k < \frac{\lambda_i(1-d)}{(d+\lambda_i)}$.
In this case D_2 will be p.d. and by using Lemma3 we have the following theorem.
Theorem2
The MAULE weighted estimator is superior to the mixed almost unbiased ridge in the MSE sense, namely, $\Delta_2$if and only if $c_1'(D_2 + b_1 b_1')^{-1} c_1 \leq 1$ for $k < \frac{\lambda_i(1-d)}{(d+\lambda_i)}$.
Now, let k be fixed for the moments. To avoid the repetition, when 0 < k <1 and $d < \frac{\lambda_i(1-k)}{(k+\lambda_i)}$, D2 will be p.d. and by using lemma 3 we have the following theorm.
Theorem 3
The MAULE is superior to the weighted mixed almost unbiased ridge estimator in the MSE sense, namely,$\Delta_2$if and only if $c_1'(D_2 + b_1 b_1')^{-1} c_1 \leq 1$ for $d < \frac{\lambda_i(1-k)}{(k+\lambda_i)}$.
As is well known to us , the values of the parameters k,d,σ and α are unknown, therefore we must estimate them as in previous studies (see Hoerl and Kennard (1970a,b) and also Liu (1993))

4. NUMERICAL EXAMPLE

To illustrate the performance of the proposed estimator in the MSE, a numerical example is given . We consider the dataset on portland cement where it has been widely analyzed in literature (Hu Yang and Jianwen Xu (2007)) and (Hu Yang et al. (2009)). By using Lemma 4 we can get Δ_i , i=1,…,3 is n.n.d if and only if each eigenvalue of $\Delta_i$is non negative.
Consider the following stochastic linear restriction: (see Hu Yang et al.,2009)
r = Rβ + e, where $e \backsim N(0, \hat{\sigma}^2_{OLS})$. $\hat{\sigma}^2_{OLS} = 5.8455$ and R=(1 , -1 , 1 , 0) , where the LSE is $\hat{\sigma} = (2.1930, 1.1533, 0.7585, 0.4863)'$
By observing Table 1, we note that the performance of the new estimator is better for different values of k and d compared with ME and this result is consistent with the theoretical results in theorem 1 and 2 .
The performance of new estimators influenced by the value of parameter k and d and this is evident in Table 2. Where in the case k is small, the estimator MAULE will be the best compared with weighted mixed almost unbiased ridge estimator and this preference decreases when the value of k is increased until to become better than MAULE when k=0.7 for all values d in this study .

Table 1 : Estimated eigenvalues of Δ_1 and for different values of d.

w=0.05			
d	0.3	0.6	0.9
$\lambda_1 (\Delta_1)$	0.854746	0.350179	0.0000012
$\lambda_2 (\Delta_1)$	-0.004184	0.000862	0.0000035
$\lambda_3 (\Delta_1)$	0.000933	-0.000143	0.0000924
$\lambda_4 (\Delta_1)$	0.000165	0.000055	0.0239074
w=0.1			
d	0.3	0.6	0.9
$\lambda_1 (\Delta_1)$	0.854647	0.350176	0.0239074
$\lambda_2 (\Delta_1)$	-0.002800	0.000699	0.0000671
$\lambda_3 (\Delta_1)$	0.000966	-0.000121	0.0000011
$\lambda_4 (\Delta_1)$	0.000154	0.000052	0.0000033
w=0.35			
d	0.3	0.6	0.9
$\lambda_1 (\Delta_1)$	0.854517	0.350173	0.0239074
$\lambda_2 (\Delta_1)$	-0.001077	0.000406	0.0000302
$\lambda_3 (\Delta_1)$	0.000918	-0.000074	0.0000008
$\lambda_4 (\Delta_1)$	0.000108	0.000037	0.0000027
w=0.75			
d	0.3	0.6	0.9
$\lambda_1 (\Delta_1)$	0.854491	0.350172	0.0239074
$\lambda_2 (\Delta_1)$	-0.000697	0.000296	0.0000199
$\lambda_3 (\Delta_1)$	0.000782	-0.000052	0.0000004
$\lambda_4 (\Delta_1)$	0.000061	0.000023	0.0000023
w=0.95			
d	0.3	0.6	0.9
$\lambda_1 (\Delta_1)$	-0.000643	0.350172	0.0239074
$\lambda_2 (\Delta_1)$	0.000047	0.000274	0.0000181
$\lambda_3 (\Delta_1)$	0.000740	-0.000047	0.0000003
$\lambda_4 (\Delta_1)$	0.854488	0.000018	0.0000023

Table 2: Estimated eiagenvalues of Δ_2 for different values of d and k

			k=0.1
d	0.3	0.6	0.9
$\lambda_1 (\Delta_2)$	35.1098	12.5983	2.64x102
$\lambda_2 (\Delta_2)$	0.0061	0.2705	5.51x10-2
$\lambda_3 (\Delta_2)$	0.2845	0.0023	4.91x10-4
$\lambda_4 (\Delta_2)$	0.8228	0.0922	1.87x10-2
			k=0.3
d	0.3	0.6	0.9
$\lambda_1 (\Delta_2)$	28.2066	5.6952	-6.90×10^{-2}
$\lambda_2 (\Delta_2)$	0.0048	0.001	-1.46×10^{-1}
$\lambda_3 (\Delta_2)$	0.2349	0.0426	-1.30×10^{-3}
$\lambda_4 (\Delta_2)$	0.6771	0.1247	-4.95x10-2
			k=0.7
d	0.3	0.6	0.9
$\lambda_1 (\Delta_2)$	-8.0138	22.5114	-3.51x10
$\lambda_2 (\Delta_2)$	-0.0013	-0.0038	-6.10×10^{-3}
$\lambda_3 (\Delta_2)$	-0.0739	-0.1922	-2.85×10^{-1}
$\lambda_4 (\Delta_2)$	-0.2107	-0.5524	-8.23×10^{-1}

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BIPOLAR COMPLEX NEUTROSOPHIC SOFT SET THEORY

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ABSTRACT

We establish the concept of bipolar complex neutrosophic soft set (BCNSS) by extending the concept of bipolar neutrosophic soft set (BNSS) from real space to the complex space. BCNSS is a hybrid structure of bipolar complex neutrosophic set (BCNS) and soft set, thus making it highly suitable for use in decision-making problems that involve positive and negative indeterminate data where the extra information provided by the phase terms of the complex numbers play a key role in determining the final decision. Based on this new concept we define the basic theoretical operations such as complement, subset, union and intersection operations. The basic properties are also verified.

Keywords: bipolar complex neutrosophic set; bipolar neutrosophic soft set ;complex neutrosophic set; neutrosophic soft set

1. INTRODUCTION

A soft set is a set-valued map defined byMolodtsov [15], to approximately describe objects usingseveral parameters. Neutrosophy [17] is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set [18] is a part of neutrosophy, handles uncertainty, indeterminacy and inconsistency. Both complex neutrosophic set [1] and neutrosophic soft set [14] are improved and generalized models of the neutrosophic set but in different spaces. Complex neutrosophic set handles the neutrosophic data which has the periodic manner, while neutrosophic soft set provides a parameterization tool to hanle the neutrosophic data.Subsequently, these uncertainty sets have been actively applied in various decision making problems to handle all types of uncertainty [3-9].

A wide variety of human decision making is based on double-sided or bipolar judgmental thinking on a positive side and a negative side. A great deal of research have been conducted to integrate the idea of bipolarity in decision making techniques by virtue of the uncertainty sets like fuzzy, intuitionistic fuzzy, complex fuzzy , neutrosophic and complex neutrosophic sets [2,10-13, 16]. Motivated by these results and as per our knowledge there is no work available on bipolar complex neutrosophicsoft set and its application.Acordingly, based on soft set theory, we introduced bipolar complex neutrosophic soft set and its operations. The results of this paper can be applied in different decision-making problems.

2. PRELIMINARIES

This section provides a brief overview of some concepts on neutrosophic sets and complex neutrosophic sets.
We begin by defining the cocepts of neutrosophic set, neutrosophic soft set and bipolar neutrosophic soft set.

Definition 2.1. Let U be a universe of discourse. A neutrosophic set N in U is defined as $N = \{< u;\ T_N(u),\ I_N(u),\ F_N(u) >: u \in U\}$, where $T_N(u), I_N(u)$and $F_N(u)$ are the truth,

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the indeterminacy and the falsity membership functions, such tha $T, I, F: U \to]^{-}0, 1^{+}[$t and $0^{-} \leq T + I + F \leq 3^{+}$.

Definition 2.2. Let U be a universe and E b set of parameters set. A pair (N, E) is called a neutrosophic soft set over U, where N is a mapping given by $N: E \to \rho(N)$.
Where $\rho(N)$ denotes the power neutrosophic set of U.

Definition 2.3. Let Ube a universe and E be a set of parameters. A bipolar neutrosophic soft set B in $U$is defined as
$B = \{< e, \{T^{+}(u), I^{+}(u), F^{+}(u), T^{-}(u), I^{-}(u), F^{-}(u)\} >: e \in E,\ u \in U\}$, where $T^{+}, I^{+}, F^{+}: U \to [0, 1]$ and $T^{-}, I^{-}, F^{-}: U \to [-1, 0]$. The positive membership degreeT^{+}, I^{+}, F^{+}denotes the truth membership, indeterminate membership and false membership of an element corresponding to a bipolar neutrosophicsoft set $B$and the negative membership degreeT^{-}, I^{-}, F^{-} denotes the truth membership,indeterminate membership and false membership of an element $u \in U$ to some implicit counter-property corresponding to a bipolar neutrosophic soft set B.

Definition 2.4. Let X be the universe. A complex neutrosophic set S in X is defined as $S = \{< x;\ T_S(x),\ I_S(x),\ F_S(x) >: x \in X\}$, where $T_S(x),\ I_S(x)$ and$F_S(x)$ are complex-valued truth, indeterminate and false membership functions and are of the form $T_S(x) = P_S(x).e^{j\mu_S(x)}$, $I_S(x) = q_S(x).e^{j\nu_S(x)}$ and $F_S(x) = r_S(x).e^{j\omega_S(x)}$.By definition, $P_S(x), q_S(x), r_S(x)$and $\mu_S(x), \nu_S(x),\ \omega_S(x)$ are, respectively, real valued and $P_S(x), q_S(x), r_S(x) \in [0,\ 1]$, such that $0^{-} \leq P_S(x) + q_S(x) + r_S(x) \leq 3^{+}$.

Definition 2.5.A bipolar complex neutrosophic set C in U is defined as:
$C = \left\{< u;\ p^{+}e^{i\mu^{+}},\ q^{+}e^{i\nu^{+}}, r^{+}e^{i\omega^{+}},\ p^{-}e^{i\mu^{-}},\ q^{-}e^{i\nu^{-}}, r^{-}e^{i\omega^{-}} >: u \in U\right\}$, where $p^{+}, q^{+}, r^{+}: U \to [0, 1]$ and $p^{-}, q^{-}, r^{-}: U \to [-1, 0]$. A bipolar complex neutrosophic number can be represented as follows.

$$C =< p^{+}e^{i\mu^{+}}, q^{+}e^{i\nu^{+}}, r^{+}e^{i\omega^{+}}, p^{-}e^{i\mu^{-}}, q^{-}e^{i\nu^{-}}, r^{-}e^{i\omega^{-}} >.$$

3. BIPOLAR COMPLEX NEUTROSOPHIC SOFT SET

Definition 3.1. Let X be a universe,A be a set of parameters. A bipolar complex neutrosophic soft set (BCNSS) (B, A) is defined as:

$(B, A) = \{< a, \{T^{+}_{B(a)}(x), I^{+}_{B(a)}(x), F^{+}_{B(a)}(x), T^{-}_{B(a)}(x), I^{-}_{B(a)}(x), F^{-}_{B(a)}(x)\} >:$
$a \in A,\ \in X\}$, where $\forall\, a \in A, \forall\, x \in X, T^{+}_{B(a)}(x) = P^{+}_{B(a)}(x)e^{2\pi i\mu^{+}_{B(a)}(x)}, I^{+}_{B(a)}(x) = q^{+}_{B(a)}(x)e^{2\pi i\nu^{+}_{B(a)}(x)}, F^{+}_{B(a)}(x) = r^{+}_{B(a)}(x)e^{2\pi i\omega^{+}_{B(a)}(x)}, T^{-}_{B(a)}(x) = P^{-}_{B(a)}(x)e^{2\pi i\mu^{-}_{B(a)}(x)}, I^{-}_{B(a)}(x) = q^{-}_{B(a)}(x)e^{2\pi i\nu^{-}_{B(a)}(x)}$, and $F^{-}_{B(a)}(x) = r^{-}_{B(a)}(x)e^{2\pi i\omega^{-}_{B(a)}(x)}$, such that :
$P^{+}, q^{+}, r^{+}, \mu^{+}, \nu^{+}, \omega^{+}: X \to [0, 1]$ and $P^{-}, q^{-}, r^{-}, \mu^{-}, \nu^{-}, \omega^{-}: X \to [-1, 0]$. The positive membership degrees T^{+}, I^{+}, F^{+}denotes, respectively the complex valued truth, indeterminacy, and falsity membership degrees of an element $x \in X$ to the property corresponding to a BCNSS (B, A), and the negative membership degrees T^{-}, I^{-}, F^{-} are to denote the complex valued truth, indeterminacy, and falsity membership degrees of an element $x \in X$ to some implicit counter-property corresponding to a BCNSS (B, A).

The following example illustrates the definition of the BCNSS.

Example 3.2. Let $X = \{x_1, x_2\}$ be a universe and $A = \{a_1, a_2\}$ be a set of parameters . Then the BCNSS (B, A) is defined as below:

(B, A)= $\{< a_1,$

$$\left\{\frac{x_1}{<0.2\, e^{2\pi i(0.5)},\ 0.1\, e^{2\pi i(0.4)},\ 0.3\, e^{2\pi i(0.8)}, -0.2\, e^{2\pi i(-0.5)}, -0.8\, e^{2\pi i(-0.7)},\ -0.1\, e^{2\pi i(-0.2)}>}\right\},$$

$$\left\{\frac{x_2}{<0.9\, e^{2\pi i(0.7)},\ 0.2\, e^{2\pi i(0.5)},\ 0.4\, e^{2\pi i(0.1)}, -0.3\, e^{2\pi i(-0.6)}, -0.1\, e^{2\pi i(-0.5)},\ -0.4\, e^{2\pi i(-0.5)}>}\right\}>,$$

$$< a_2, \left\{\frac{x_1}{<0.5\, e^{2\pi i(0.6)},\ 0.4\, e^{2\pi i(0.3)},\ 0.1\, e^{2\pi i(0.5)}, -0.2\, e^{2\pi i(-0.7)}, -0.3\, e^{2\pi i(-0.4)},\ -0.2\, e^{2\pi i(-0.6)}>}\right\},$$

$$\left\{\frac{x_2}{<0.8\, e^{2\pi i(0.4)},\ 0.2\, e^{2\pi i(0.4)},\ 0.7\, e^{2\pi i(0.9)}, -0.9\, e^{2\pi i(-0.4)}, -0.8\, e^{2\pi i(-0.2)},\ -0.7\, e^{2\pi i(-0.5)}>}\right\}>\}$$

Now we put forward the definition of the empty BCNSS and the definition of the absolute BCNSS.

Definition 3.3. Let (B, A) be a BCNSS over X. Then (B, A) is said to be empty BCNSS denoted by $B_\emptyset$, if $T^+_{B(a)}(x) = 0, I^+_{B(a)}(x) = 1, F^+_{B(a)}(x) = 1$ and $T^-_{B(a)}(x) = 0, I^-_{B(a)}(x) = -1, F^-_{B(a)}(x) = -1, \forall a \in A, \forall x \in X$and defined as :
$(B_\emptyset, A) = \{< a, \{0, 1, 1, 0, -1, -1\} >: a \in A, x \in X\}$.

Definition 3.4. Let (B, A) be a BCNSS over X. Then (B, A) is said to be absolute BCNSS denoted by B_X, if $T^+_{B(a)}(x) = 1, I^+_{B(a)}(x) = 0, F^+_{B(a)}(x) = 0$ and $T^-_{B(a)}(x) = -1, I^-_{B(a)}(x) = 0, F^-_{B(a)}(x) = 0, \forall a \in A, \forall x \in X$and defined as :
$(B_X, A) = \{< a, \{1, 0, 0, -1,0,0\} >: a \in A, x \in X\}$.

In the following , we introduce the concept of the complement of the BCNSS.

Definition 3.5. Let X be a universe of discourse and (B, A) be a BCNSS on X.
The complement of (B, A) is denoted by $(B, A)^c = (B^c, A)$ and is defined as:
$(B, A)^c = \{< a, \{T^+_{B^c(a)}(x), I^+_{B^c(a)}(x), F^+_{B^c(a)}(x), T^-_{B^c(a)}(x), I^-_{B^c(a)}(x), F^-_{B^c(a)}(x)\} >:$ $a \in A,\ x \in X\}$, where

$$T^+_{B^c(a)}(x) = P^+_{B^c(a)}(x) e^{2\pi i \mu^+_{B^c(a)}(x)} = r^+_{B(a)}(x) e^{2\pi i \omega^+_{B(a)}(x)},$$
$$I^+_{B^c(a)}(x) = q^+_{B^c(a)}(x) e^{2\pi i v^+_{B^c(a)}(x)} = \left(1 - q^+_{B(a)}(x)\right) e^{2\pi i (1 - v^+_{B(a)}(x))},$$
$$F^+_{B^c(a)}(x) = r^+_{B^c(a)}(x) e^{2\pi i \omega^+_{B^c(a)}(x)} = P^+_{B(a)}(x) e^{2\pi i \mu^+_{B(a)}(x)},$$
$$T^-_{B^c(a)}(x) = P^-_{B^c(a)}(x) e^{2\pi i \mu^-_{B^c(a)}(x)} = r^-_{B(a)}(x) e^{2\pi i \omega^-_{B(a)}(x)},$$
$$I^-_{B^c(a)}(x) = q^-_{B^c(a)}(x) e^{2\pi i v^-_{B^c(a)}(x)} = \left(-1 - q^-_{B(a)}(x)\right) e^{2\pi i (-1 - v^-_{B(a)}(x))},$$
$$F^-_{B^c(a)}(x) = r^-_{B^c(a)}(x) e^{2\pi i \omega^-_{B^c(a)}(x)} = P^-_{B(a)}(x) e^{2\pi i \mu^-_{B(a)}(x)}.$$

Example 3.6. Consider Example 3.2. By Definition 3.5, we obtain the complement of the BCNSS (B, A) given by

$(B, A)^c$= $\{< a_1,$

$$\left\{\frac{x_1}{<0.3\, e^{2\pi i(0.8)},\ 0.9\, e^{2\pi i(0.6)},\ 0.2\, e^{2\pi i(0.5)}, -0.1\, e^{2\pi i(-0.2)}, -0.2\, e^{2\pi i(-0.3)},\ -0.2\, e^{2\pi i(-0.5)}>}\right\},$$

$$\left\{\frac{x_2}{<0.4\, e^{2\pi i(0.1)},\ 0.8\, e^{2\pi i(0.5)},\ 0.9\, e^{2\pi i(0.7)}, -0.4\, e^{2\pi i(-0.5)}, -0.9\, e^{2\pi i(-0.5)},\ -0.3\, e^{2\pi i(-0.6)}>}\right\}$$
$>,$

$$< a_2, \left\{\frac{x_1}{<0.1\, e^{2\pi i(0.5)},\ 0.6\, e^{2\pi i(0.2)},\ 0.5\, e^{2\pi i(0.6)}, -0.2\, e^{2\pi i(-0.6)}, -0.7\, e^{2\pi i(-0.6)},\ -0.2\, e^{2\pi i(-0.7)}>}\right\},$$

$$\left\{\frac{x_2}{< 0.7\, e^{2\pi i(0.9)},\ 0.8\, e^{2\pi i(0.6)},\ 0.8\, e^{2\pi i(0.4)}, -0.7\, e^{2\pi i(-0.5)}, -0.2\, e^{2\pi i(-0.8)},\ -0.9\, e^{2\pi i(-0.4)} >}\right\}$$

$>\}$

Proposition 3.7. *If* (B, A)*is a BCNSS over the universe* X. *Then*$((B,\ A)^c)^c = (B,\ A)$.

Proof. The proof is straitforward from Definition 3.5.□

Now, we establish the definitions of the subset, union and intersectionof two BCNSSs.

Definition3.8. For two BCNSSs (B, A) and $(B',\ A')$ over X . A BCNSS (B, A) is contained in the BCNSS $(B',\ A')$, denoted as $(B, A) \sqsubseteq (B',\ A')$if and only if:

(1) $A \subseteq A'$, and (2) $\forall a \in A,\ \forall x \in X,\ P^+_{B(a)}(x) \le P^+_{B'(a)}(x),\ q^+_{B(a)}(x) \ge q^+_{B'(a)}(x), r^+_{B(a)}(x) \ge r^+_{B'(a)}(x),\ \mu^+_{B(a)}(x) \le \mu^+_{B'(a)}(x), \nu^+_{B(a)}(x) \ge \nu^+_{B'(a)}(x),$ $\omega^+_{B(a)}(x) \ge \omega^+_{B'(a)}(x)$ and $P^-_{B(a)}(x) \ge P^-_{B'(a)}(x), q^-_{B(a)}(x) \le q^-_{B'(a)}(x),\ r^-_{B(a)}(x) \le r^-_{B'(a)}(x), \mu^-_{B(a)}(x) \ge \mu^-_{B'(a)}(x), \nu^-_{B(a)}(x) \le \nu^-_{B'(a)}(x), \omega^-_{B(a)}(x) \le \omega^-_{B'(a)}(x).$

Definition 3.9. Lex$X$ be a universe. The union (intersection) of two BCNSSs (B, A) and $(B',\ A')$denoted as $(B, A) \sqcup(\sqcap)(B^{'},\ A^{'})$is a BCNSS (C, D), where $D = A\ \cup\ A'$ and $\forall \epsilon \in D$, $\forall x\ \in X$,

$$T^+_{C(\epsilon)} = \begin{cases} P^+_{B(\epsilon)}(x)e^{2\pi i \mu^+_{B(\epsilon)}(x)} if \epsilon \in\ A - A' \\ P^+_{B'(\epsilon)}(x)e^{2\pi i \mu^+_{B'(\epsilon)}(x)} if \epsilon\ \in\ A' - A \\ \left(P^+_{B(\epsilon)}(x) \vee (\wedge)\ P^+_{B'(\epsilon)}(x)\right).e^{2\pi i(\mu^+_{B(\epsilon)}(x)\ \vee(\wedge) \mu^+_{B'(\epsilon)}(x))} if\ \epsilon \in\ A \cap\ A' \end{cases}$$

$$I^+_{C(\epsilon)} = \begin{cases} q^+_{B(\epsilon)}(x)e^{2\pi i \nu^+_{B(\epsilon)}(x)} if \epsilon \in\ A - A' \\ q^+_{B'(\epsilon)}(x)e^{2\pi i \nu^+_{B'(\epsilon)}(x)} if \epsilon\ \in\ A' - A \\ \left(q^+_{B(\epsilon)}(x) \wedge (\vee)\ q^+_{B'(\epsilon)}(x)\right).e^{2\pi i(\nu^+_{B(\epsilon)}(x)\ \wedge(\vee) \nu^+_{B'(\epsilon)}(x))} if\ \epsilon \in\ A \cap\ A' \end{cases}$$

$$F^+_{C(\epsilon)} = \begin{cases} r^+_{B(\epsilon)}(x)e^{2\pi i \omega^+_{B(\epsilon)}(x)} if \epsilon \in\ A - A' \\ r^+_{B'(\epsilon)}(x)e^{2\pi i \omega^+_{B'(\epsilon)}(x)} if \epsilon\ \in\ A' - A \\ \left(r^+_{B(\epsilon)}(x) \wedge (\vee)\ r^+_{B'(\epsilon)}(x)\right).e^{2\pi i(\omega^+_{B(\epsilon)}(x)\ \wedge(\vee) \omega^+_{B'(\epsilon)}(x))} if\ \epsilon \in\ A \cap\ A' \end{cases}$$

$$T^-_{C(\epsilon)} = \begin{cases} P^-_{B(\epsilon)}(x)e^{2\pi i \mu^-_{B(\epsilon)}(x)} if \epsilon \in\ A - A' \\ P^-_{B'(\epsilon)}(x)e^{2\pi i \mu^-_{B'(\epsilon)}(x)} if \epsilon\ \in\ A' - A \\ \left(P^-_{B(\epsilon)}(x) \wedge (\vee)\ P^-_{B'(\epsilon)}(x)\right).e^{2\pi i(\mu^-_{B(\epsilon)}(x) \wedge (\vee \mu^-_{B'(\epsilon)}(x))} if\ \epsilon \in\ A \cap\ A' \end{cases}$$

$$I_{C(\epsilon)}^{-} = \begin{cases} q_{B(\epsilon)}^{-}(x)e^{2\pi i v_{B(\epsilon)}^{-}(x)} if \epsilon \in A - A' \\ q_{B'(\epsilon)}^{-}(x)e^{2\pi i v_{B'(\epsilon)}^{-}(x)} if \epsilon \in A' - A \\ \left(q_{B(\epsilon)}^{-}(x) \vee (\wedge)\, q_{B'(\epsilon)}^{-}(x)\right). e^{2\pi i (v_{B(\epsilon)}^{-}(x) \vee (\wedge) v_{B'(\epsilon)}^{-}(x))} if\ \epsilon \in A \cap A' \end{cases}$$

$$F_{C(\epsilon)}^{-} = \begin{cases} r_{B(\epsilon)}^{-}(x)e^{2\pi i \omega_{B(\epsilon)}^{-}(x)} if \epsilon \in A - A' \\ r_{B'(\epsilon)}^{-}(x)e^{2\pi i \omega_{B'(\epsilon)}^{-}(x)} if \epsilon \in A' - A \\ \left(r_{B(\epsilon)}^{-}(x) \vee (\wedge)\, r_{B'(\epsilon)}^{-}(x)\right). e^{2\pi i (\omega_{B(\epsilon)}^{-}(x) \vee (\wedge) \omega_{B'(\epsilon)}^{-}(x))} if\ \epsilon \in A \cap A' \end{cases}$$

Proposition 3.10. *The following properties are hold for the BCNSSs* (B, A) ,$(B',\ A')$*and* $(B'',\ A'')$.

(1)$(B_{\emptyset}, A)^c = (B_X, A)$,
(2)$(B_X, A)^c = (B_{\emptyset}, A)$,
(3) $(B, A) \sqcup (B_{\emptyset}, A) = (B, A)$,
(4)$(B, A) \sqcup (B_X, A) = (B_X, A)$,
(5)$(B, A) \sqcap (B_{\emptyset}, A) = (B_{\emptyset}, A)$,
(6)$(B, A) \sqcap (B_X, A) = (B, A)$,
(7)$(B, A) \sqcup (B',\ A') = (B',\ A') \sqcup (B, A)$,
(8)$(B, A) \sqcap (B',\ A') = (B',\ A') \sqcap (B, A)$,
(9)$(B, A) \sqcup ((B',\ A') \sqcup (B'',\ A'')) = ((B, A) \sqcup (B',\ A')) \sqcup (B'',\ A'')$,
(10)$(B, A) \sqcap ((B',\ A') \sqcap (B'',\ A'')) = ((B, A) \sqcap (B',\ A')) \sqcap (B'',\ A'')$,
(11)$(B, A) \sqcup ((B',\ A') \sqcap (B'',\ A'')) = ((B, A) \sqcup (B',\ A')) \sqcap \left((B, A) \sqcup (B'',\ A'')\right)$,
(12)$(B, A) \sqcap ((B',\ A') \sqcup (B'',\ A'')) = ((B, A) \sqcap (B',\ A')) \sqcup \left((B, A) \sqcap (B'',\ A'')\right)$,
(13) $((B, A) \sqcup ((B',\ A'))^c = (B, A)^c \sqcap (B',\ A')^c$,
(14) $((B, A) \sqcap ((B',\ A'))^c = (B, A)^c \sqcup (B',\ A')^c$.

8. CONCLUSION

We established the concept of bipolar complex neutrosophic soft set (BCNSS) as a generalization of both bipolar complex neutrosophic set and bipolar neutrosophic soft set. Some essential operations such as complement, subset, union and intersection with their properties are defined and verified. BCNSS seems to be a promising new concept, paving the way toward numerous possibilities for future research. We intend to investigate this concept further to develop some real applications.

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APPLICATION OF RESIDUAL POWER SERIES METHOD FOR SOLVING NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS IN FRACTIONAL SENSE

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ABSTRACT

This work aims to develop a reliable approximation tool to solve the nonlinear fractional integro-differential equations that include a Fredholm operator under Caputo fractional concept. The proposed technique is mainly based on the use of residual power series method combining the generalized Taylor's series and residual error function. This technique can be applied directly to the solutions of nonlinear phenomena without the need for linearity or set any limitations on the problem’s nature or the number of grid points. To verify the accuracy and applicability of this technique, numerical example is performed. The results are carried out using the Mathematica software package, which indicate that the method is straightforward, and convenient for approximate rough solutions for nonlinear fractional models arising in various fields of applied science.

Keywords: Caputo fractional derivative; residual power series method; analytical solution; Fredholm integro-differential equations.

1. INTRODUCTION

The fractional differentiation and integration theory is indeed a generalization of ordinary calculus theory that deals with differentiation and integration to an arbitrary order, which is utilized to describe various real-world phenomena arising in natural sciences, applied mathematics, and engineering fields [1-3]. Many mathematical forms of real-world issues contain nonlinear fractional integro-differential equations (FIDEs). Since most fractional differential and integro-differential equations cannot be solved analytically, thus it is necessary to find an accurate numerical and analytical methods to deal with the complexity of fractional operators involving such equations. Anyhow, in recent times, many experts have devoted their interest in finding solutions of the fractional integro-differential equations utilizing different analytic-numeric methods. For instance, Adomian decomposition method, variational iteration method,chebyshev wavelet, Legendre ploynomail method, multistep approach, and reproducing kernel method [4-14].

The basic goal of the present work is to introduce a recent analytic-numeric method based on the use of residual power series technique for obtaining the numerical approximate solution for a class of nonlinear fractional Fredholm integro-differential equations in the form

$$\mathcal{D}_{a^+}^{\beta}\varphi(t) + \int_0^1 h(t,s)(\varphi(s))^r\, ds = f(t), 0 < \beta \leq 1, r \geq 2, \tag{1}$$

with the initial condition

$$\varphi(0) = \varphi_0 \in \mathbb{R}, \tag{2}$$

where $\mathcal{D}_{a^+}^{\beta}$ denotes the Caputo fractional derivative, $f(t)$ and $h(t,s)$ are smooth functions. Here, $\varphi(t)$ is unknown analytic function to be determined.

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The residual power series (RPS) method is a recent analytic-numeric treatment method based on power series expansion was proposed by Abu Arqub in [15] to provide analytical series solutions of first and second-order fuzzy differential equations. The method is easy and applicable to find the series solutions for several types of the non-linear differential equation and integrodifferential equations of fractional order without being linearized, discretized, or exposed to perturbation. The RPS method has been successfully applied to solve linear and non-linear ordinary, partial and fuzzy differential equations for more details, see [16-26].

The rest of the current paper is as follow: In next section, we introduce some essential preliminaries related to fractional calculus and fractional power series representations. In Section 3, we illustrate the solution methodology by using the RPS technique. In Section 4, illustrative problems are given to demonstrate the simplicity, accuracy, and performance of the present method. Finally, we give concluding remark in Section 5.

2. PRELIMINARIES

In this section, we recall some definitions and basic results concerning fractional calculus and fractional power series representations [27-34].

Definition 2.1: The Riemann-Liouville fractional integral operator of order β, over the interval $[a,b]$ for a function $\varphi \in L_1[a,b]$ is defined by

$$\mathcal{J}_{a^+}^{\beta}\varphi(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\beta}}\, d\tau, 0 < \tau < t, \beta > 0, \\ \varphi(t), \qquad \beta = 0. \end{cases}.$$

Definition 2.2: For $\beta > 0, a, t, \beta \in \mathbb{R}$. Then the following fractional derivative of order β

$$\mathcal{D}_{a^+}^{\beta}\varphi(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}\, d\tau,$$

$n-1 < \beta < n$ for $n \in \mathbb{N}$, is referred to the Caputo fractional differential operator of order β. In case $\beta = n$, then $\mathcal{D}_{a^+}^{\beta}\varphi(t) = \frac{d^n}{dt^n}\varphi(t)$.

The following are some interesting properties of the operator $\mathcal{D}_{a^+}^{\beta}$:

- For any constant $c \in \mathbb{R}$, then $\mathcal{D}_{a^+}^{\beta}c = 0$,
- $\mathcal{D}_{a^+}^{\beta}(t-a)^q = \begin{cases} \frac{\Gamma(q+1)}{\Gamma(q+1-\beta)}(t-a)^{q-\beta}, n-1 < \beta \leq n, q > n-1, n \in \mathbb{N}, q \in \mathbb{R} \\ 0, \quad \text{otherwise.} \end{cases}$
- $\mathcal{D}_{a^+}^{\beta}\mathcal{J}_{a^+}^{\beta}\varphi(t) = \varphi(t)$,
- $\mathcal{J}_{a^+}^{\beta}\mathcal{D}_{a^+}^{\beta}\varphi(t) = \varphi(t) - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(a^+)}{k!}(t-a)^k$.

Definition 2.3: A fractional power series (FPS) representation at $t = a$ has the following form

$$\sum_{m=0}^{\infty} c_m(t-a)^{m\beta} = c_0 + c_1(t-a)^{\beta} + c_2(t-a)^{2\beta} + \cdots,$$

where $0 \leq n-1 < \beta \leq n$ and $t \geq a$, and c_m's are the coefficients of the series.

Theorem 2.1: *Suppose that $\varphi(t)$ has the following FPS representation at $t = a$*

$$\varphi(t) = \sum_{m=0}^{\infty} c_m(t-a)^{m\beta},$$

where $n-1<\beta\le n, a<t<a+R, \varphi(t)\in C[a,a+R)$ *and* $\mathcal{D}_{a^+}^{m\beta}\varphi(t)\in C(a,a+R)$ *for* $m=0,1,2,\ldots$, *then the coefficients* c_m *will be in the form* $c_m=\frac{\mathcal{D}_{a^+}^{m\beta}\varphi(a)}{\Gamma(m\beta+1)}$ *such that* $\mathcal{D}_{a^+}^{m\beta}=\mathcal{D}_{a^+}^{\beta}\cdot\mathcal{D}_{a^+}^{\beta}\cdot\ldots\cdot\mathcal{D}_{a^+}^{\beta}$ *(m-times).*

3. CONSTRUCTION SOLUTION BY RPS ALGORITHM

The purpose of this section is to construct FPS solution for non-linear fractional Fredholm integro-differential equations (1) and (2) by substitute its FPS expansion among its truncated residual function.The RPS algorithm proposed the solution of Eqs. (1) and (2) about $a=0$ has the following FPS expansion:

$$\varphi(t)=\sum_{m=0}^{\infty}c_m\frac{t^{m\beta}}{\Gamma(m\beta+1)}. \tag{3}$$

For obtaining the approximate values of Eq. (3), consider the following k^{th}-FPS approximate solution

$$\varphi_k(t)=\sum_{m=0}^{k}c_m\frac{t^{m\beta}}{\Gamma(m\beta+1)}. \tag{4}$$

Clearly, if $=0, \varphi(0)=\varphi_0$. So, the expansion (4) can be written as

$$\varphi_k(t)=\varphi_0+\sum_{m=1}^{k}c_m\frac{t^{m\beta}}{\Gamma(m\beta+1)}. \tag{5}$$

Define the so-called the residual function for Eqs. (1) and (2) as follows:

$$Res(t)=\mathcal{D}_{0^+}^{\beta}\varphi(t)+\int_0^1 h(t,s)(\varphi(s))^r\,ds-f(t), \tag{6}$$

and the following k^{th}-residual function

$$Res_k(t)=\mathcal{D}_{0^+}^{\beta}\varphi_k(t)+\int_0^1 h(t,s)(\varphi_k(s))^r\,ds-f(t), \tag{7}$$

As in [21-25], some useful properties of residual function

- $\lim_{k\to\infty}Res_k(t)=Res(t)=0$, for each $t\in(0,1)$.
- $\mathcal{D}_{0^+}^{m\beta}Res(0)=\mathcal{D}_{0^+}^{m\beta}Res_k(0)=0$ for each $m=0,1,2,\ldots,k$.

For obtaining the coefficients c_m, $m=0,1,2,\ldots,k$, solve the solution of the following relation:

$$\mathcal{D}_{0^+}^{(k-1)\beta}Res_k(0)=0, \qquad k=1,2,3,\ldots \tag{8}$$

4. NUMERICAL EXAMPLES

This section aims to test two nonlinear FFIDEs in order to demonstrate the efficiency, accuracy, and applicability of the present novel approach. Here, all necessary calculations and analyses are done using Mathematica 11.

Example 4.1: Consider the following nonlinear fractional Fredholm integro-differential equation

$$\mathcal{D}_{a^+}^{\beta}\varphi(t) + \int_0^1 st^5\varphi(s)^3\,ds = \frac{1}{9}(2e^3 + 1)t^5 + e^t, 0 < \beta \leq 1, \tag{9}$$

with the initial condition

$$\varphi(0) = 1. \tag{10}$$

Here, the exact solution at$\beta = 1$ is given by $\varphi(t) = e^t$.

Using the RPS algorithm, The k-th residual function $Res_k(\mathrm{t})$ is given by

$$Res_k(t) = \mathcal{D}_{a^+}^{\beta}\varphi(t) + \int_0^1 st^5\varphi(s)^3\,ds = \frac{1}{9}\left(2e^3 + 1\right)t^5 + e^t, \tag{11}$$

where $\varphi_k(t)$ has the form

$$\varphi_k(t) = 1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta + 1)}.$$

Consequently,

$$Res_k(t) = \mathcal{D}_{a^+}^{\beta}\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta + 1)}\right) + \int_0^1 st^5\left(1 + \sum_{m=1}^{k} c_m \frac{s^{m\beta}}{\Gamma(m\beta + 1)}\right)^3 ds$$
$$- \left(\frac{1}{9}\left(2e^3 + 1\right)t^5 + e^t\right).$$

The absolute errors are listed in Table 1. The results obtained by the RPS method show that the exact solutions are in good agreement with approximate solutions at $\beta = 1, n = 6$ and step size 0.2. While Table 2 show approximate solutions at different values of β such that $\beta \in \{1, 0.9, 0.8, 0.7\}$ with step size 0.16. From the table, one can be found that the RPS method provides us with an accurate approximate solution, which is in good agreement with the exact solutions for all values of t in $[0,1]$.

Table 1: Absolute error for Example 4.1 at $\beta = 1$.

t	Exact Sol.	Approximate Sol.	Absolute Error
0.2	1.221402758160169	1.2214027555555556	2.60461×10^{-9}
0.4	1.491824697641270	1.4918243555555555	3.42085×10^{-7}
0.6	1.822118800390509	1.8221128000000000	6.00039×10^{-6}
0.8	2.225540928492468	2.2254947555555558	4.61729×10^{-5}

Table 2: Numerical results for Example 4.1 for different values of β.

t	6th RPS solution			
	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.16	1.1735108704	1.2236588706	1.2896293585	1.3781327965
0.32	1.3771276933	1.4620068483	1.5701199217	1.7112052001
0.48	1.6160731635	1.7354009578	1.8854925107	2.0791905574
0.64	1.8964714019	2.0527277406	2.2480109650	2.4982975959
0.80	2.2254947555	2.4227191207	2.6681456160	2.9806767189
0.96	2.6115273760	2.8549680620	3.1566853797	3.5379276118

CONCLUDING REMARKS

The present paper aims to solve a class of nonlinear fractional Fredholm integro-differential equations of order $\beta: 0 < \beta \leq 1$, based on the use of RPS algorithm. The solution methodology depends on the constructing of the residual function and applying the generalized Taylor formula under the Caputo fractional derivative. The proposed algorithm provides the solutions in the form of rapidly convergent series with no need linearization, limitation on the problem's nature, sort of classification or perturbation. Numerical results are performed by Mathematica 10. The results demonstrate the accuracy, efficiency and the capability of the present method.

Therefore, the RPS algorithm is reliable, effective, simple, straightforward tool for handling a wide range of nonlinear fractional integro-differential equations.

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STUDYING THE EFFECT OF SOME VARIABLES ON THE ECONOMIC GROWTH USING LATENT ROOTS METHOD

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ABSTRACT

Different kinds of estimators have been proposed as an alternative to the ordinary least squares for estimating the coefficients of the multiple linear regression model in the presence of multicollinearity. We estimated the parameters of this linear model by two methods: the least squares and the latent roots method. A comparison between these two methods is given through the application of the economic growth data of the UAE to study the effect of the population size, exchange rate, total exports and the total imports on the economic growth. It is shownthat all the explanatory variables using the latent roots method have an effect on the economic growth and this effect is significant, whereas these variables are not significant using the least squares method.

Keywords: Regression; Multicollinearity; Least Squares; Correlation Matrix; Eigen Values; Eigen Vectors ;Latent Roots

1. MULTIPLE LINEAR REGRESSIONMODEL

The study of any particular phenomenon requires the identification of the factors influencing this phenomenon and the formulation of the relationship between these factors in the form of a model that expresses them. This model may be represented by one or several equations. In terms of a single equation, it may be simple or may be multiple. Common forms of use include a linear one that takes a mathematical form in writing and including more than one explanatory variable. This model will be used in this research and the general formula for this model is (Yan & Gang Su, 2009):

$$\underline{y} = X\underline{\beta} + \underline{u} \qquad \ldots.(1)$$

Where:

$\underline{y}$: is (n×1) vector of observations of the response variable.

X: is (n×k); k=p+1,matrix of observations of the explanatory variables whose first column contains the values of one.

$\underline{\beta}$:is (k×1) vector of the parameters to be estimated.

$\underline{u}$:is (n×1) vector of random errors.

In order to estimate the parameters of the model and to ensure that the estimations have desirable properties, there are certain hypotheses that must be met (Chatterjee& Price, 2000).

2. LEAST SQUARES METHOD

This method is one of the most widely used methods for estimating the parameters of the linear regression model. The least squares estimate of the regression parameters in this method are(Kutner et al., 2005):

$$\underline{\hat{\beta}}_{OLS} = (X'X)^{-1}\left(X'\underline{y}\right) \qquad \ldots.(2)$$

Here are the properties of this method(Draper & Smith, 1981)(Fisher, 1981&Mason) and (Gunst& Mason, 1980):

1. Linearity :the estimated parameters in this method are linear in terms of the response variables :

   $$\underline{\hat{\beta}}_{OLS} = (X'X)^{-1}\left(X'\underline{y}\right) = [(X'X)^{-1}X']\underline{y}$$

2. Unbiased :That is, the expected value of the estimated parameters is equal to its real value:

$E\left[\underline{\hat{\beta}}_{OLS}\right] = \underline{\beta}$

3. Variance: The variance of the estimated parameters is minimum ,where

$$Var\left(\underline{\hat{\beta}}_{OLS}\right) = (X'X)^{-1}\sigma^2 \qquad \ldots.(3)$$

and$\sigma^2 = \frac{\underline{y}'\underline{y}-\underline{\hat{\beta}}'_{OLS}X'\underline{y}}{n-p-1}$

3. THE CONCEPT OF THE MULTICOLLINEARITY IN THE REGRESSION MODEL

Multicollinearity is one of the problems that occur in many cases due to the existence of a relationship between the explanatory variables. The existence of the complete multicollinearity between the variables leads to making the matrix $(X'X)$not of full rank, ie, its determinant is zero. Thus, it is difficult to find the inverse of this matrix, Which means that the regression parameters can not be estimated using the Least Squares method. The existence of an incomplete but powerful multicollinearity leads to the amplification of the variance and thus the acquisition of inaccurate capabilities(Dounald, 1987) and (Chatterjee et al., 2000).

4. DETECTING MULTICOLLINEARITY IN THE REGRESSION MODEL

Multicollinearitycan be detected by many methods(Draper & Smith, 1981),(Fisher, 1981&Mason) and (Gunst& Mason, 1980):

1. The correlation coefficients matrix:
2. Determinant of matrix:
3. Latent values for $(X'X)$ matrix:

5. SOLUTION OF MULTICOLLINEARITY

There are several methods proposed to minimize the effect of multicollinearity, Such as (Fisher & Mason, 1981):

1. Delete the explanatory variables that are associated with other variables in order to get rid of the effects of this link and this deletion process according to certain criteria proposed to delete the specific variables.
2. Add new data to the original data.
3. Use biased estimation methods.

6. LATENT ROOTS METHOD

This method was proposed in 1973 by Hawkins, the idea of this method is to find the latent roots of the correlation matrix and then to exclude the roots that are not important in the prediction process. The following is a detailed explanation of this method (Mason, 1986):Correlation matrix is obtained by multiplying the transpose matrix (A) and the same matrix ie:

$R = A'A$

Where:

A: Is the standardized information matrix which contains the standardized values of the response variable and the standardized values of the explanatory variables:

$A = [\underline{y}^* \quad X^*]$

R: the Correlation matrix between all variable, it is defined as follows:

$$R = \begin{bmatrix} 1 & r_{y1} & r_{y2} & r_{y3} & \cdots & r_{yp} \\ r_{1y} & 1 & r_{12} & r_{13} & \cdots & r_{1p} \\ r_{2y} & r_{21} & 1 & r_{23} & \cdots & r_{2p} \\ r_{3y} & r_{31} & r_{32} & 1 & \ldots & r_{3p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{py} & r_{p1} & r_{p2} & r_{p3} & \ldots & 1 \end{bmatrix}$$

Latent roots:latent values and latent vectors, are obtained according to the following formula:

$$\Lambda = \begin{bmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0p} \\ \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{p0} & \gamma_{p1} & \cdots & \gamma_{pp} \end{bmatrix} = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0p} \\ \underline{\gamma}_0 & \underline{\gamma}_1 & \cdots & \underline{\gamma}_p \end{bmatrix}$$

The estimation of the regression parameters vector by Least Squares Method that based on the latent roots, are as follows:

$$\underline{\hat{\beta}}_{OLS} = -\frac{\sum_{j=0}^{p} \frac{\gamma_{oj}\underline{\gamma}_j}{\lambda_j}}{\sum_{j=0}^{p} \frac{\gamma_{oj}^2}{\lambda_j}} \quad \dots .(4)$$

To find the Latent Root Estimators, all the values and vectors that are not significant in the prediction are deleted from the equation (4), the roots that meet the following conditions are deleted:

$$\lambda_j < 1 \qquad and \quad |\gamma_{0j}| < 0.5 \quad for \quad j = 0,1,2,\dots,p$$

The remaining latent roots are less than or equal to p, denoted by q, and estimated by the Latent Roots Method are as follows:

$$\underline{\hat{\beta}}_{LR} = -\frac{\sum_{j=0}^{q} \frac{\gamma_{oj}\underline{\gamma}_j}{\lambda_j}}{\sum_{j=0}^{q} \frac{\gamma_{oj}^2}{\lambda_j}} \quad \dots .(5)$$

LatentRoot estimators have the following properties:

1. Bias: The Latent Root estimator is biased and its bias is:

$$\text{Bais}\left(\underline{\hat{\beta}}_{LR}\right) = \frac{\sum_{j=q+1}^{p} \frac{\gamma_{oj}\underline{\gamma}_j}{\lambda_j}}{\sum_{j=q+1}^{p} \frac{\gamma_{oj}^2}{\lambda_j}} \quad \dots .(6)$$

2. The variance: The variance of the Least Squares estimators in terms of latent roots is:

$$\text{Var}\left(\underline{\hat{\beta}}_{OLS}\right) = \sigma_{OLS}^2 \left[\sum_{j=0}^{p} \frac{\underline{\gamma}_j\underline{\gamma}_j'}{\lambda_j} - \frac{\left(\sum_{j=0}^{p} \frac{\gamma_{oj}\underline{\gamma}_j}{\lambda_j}\right)\left(\sum_{j=0}^{p} \frac{\gamma_{oj}\underline{\gamma}_j'}{\lambda_j}\right)}{\sum_{j=0}^{p} \frac{\gamma_{oj}^2}{\lambda_j}} \right] \quad \dots .(7)$$

The variance of the Latent Root estimators is :

$$\text{Var}\left(\underline{\hat{\beta}}_{LR}\right) = \sigma_{LR}^2 \left[\sum_{j=0}^{q} \frac{\underline{\gamma}_j\underline{\gamma}_j'}{\lambda_j} - \frac{\left(\sum_{j=0}^{q} \frac{\gamma_{oj}\underline{\gamma}_j}{\lambda_j}\right)\left(\sum_{j=0}^{q} \frac{\gamma_{oj}\underline{\gamma}_j'}{\lambda_j}\right)}{\sum_{j=0}^{q} \frac{\gamma_{oj}^2}{\lambda_j}} \right] \quad \dots .(8)$$

The variance of the ith estimator is:

$$\text{Var}\left(\underline{\hat{\beta}}_{iLR}\right) = \sigma_{LR}^2 \left[\sum_{j=0}^{q} \frac{\gamma_{ij}^2}{\lambda_j} - \frac{\left(\sum_{j=0}^{q} \frac{\gamma_{oj}\gamma_{ij}}{\lambda_j}\right)^2}{\sum_{j=0}^{q} \frac{\gamma_{ij}^2}{\lambda_j}} \right]$$

3. Mean Squares Error: Since the Latent Roots estimator is biased, this makes the mean squares error as follows:

$$\text{MSE}\left(\underline{\hat{\beta}}_{LR}\right) = \sigma_{LR}^2 \sum_{i=1}^{p} \left[\sum_{j=0}^{q} \frac{\gamma_{ij}^2}{\lambda_j} - \frac{\left(\sum_{j=0}^{q} \frac{\gamma_{oj}\gamma_{ij}}{\lambda_j}\right)^2}{\sum_{j=0}^{q} \frac{\gamma_{oj}^2}{\lambda_j}} \right] + \sum_{i=1}^{p} \left[\frac{\sum_{j=q+1}^{p} \frac{\gamma_{oj}\gamma_{ij}}{\lambda_j}}{\sum_{j=q+1}^{p} \frac{\gamma_{oj}^2}{\lambda_j}} \right]^2$$

Basilevsky in 1994 suggested a approximation of the mean squares error for the Latent Roots estimator as:

$$MSE\left(\underline{\hat{\beta}}_{LR}\right) \cong \sigma^2_{LR}\sum_{i=1}^{p}\left[\sum_{j=0}^{q}\frac{\gamma^2_{ij}}{\lambda_j} - \frac{\left(\sum_{j=0}^{q}\frac{\gamma_{0j}\gamma_{ij}}{\lambda_j}\right)^2}{\sum_{j=0}^{q}\frac{\gamma^2_{0j}}{\lambda_j}}\right] + \left(\underline{\gamma}'_0\underline{\hat{\beta}}_{LR}\right)^2 \quad \ldots.(9)$$

7. APPLICATION PART

A comparisonbetween the least squares and the latent roots regression methods is given through the application of the economic growth data of the UAE (Alsaffar, 2016).The data represent the Economic growth (Y) and four explanatory variables for the period (1999-2008). The explanatory variables are: (X_1) Population size;(X_2) Exchange rate; (X_3) Total export; (X_4) Total imports.

7.1The correlation matrix is given in table (1) below:

Table (1): The simple correlation coefficient between the explanatory variables and the independent variable

	Y	**X_1**	**X_2**	**X_3**	**X_4**
Y	1	0.9461	0.9942	0.99	0.9847
X_1	0.9461	1	0.9504	0.9269	0.9057
X_2	0.9942	0.9504	1	0.9954	0.9919
X_3	0.99	0.9269	0.9954	1	0.9935
X_4	0.9847	0.9057	0.9919	0.9935	1

7.2 The Latent Roots and Latent Vectors of the Correlation Matrix were found using a program written in the MATLAB language

7.3 Table 2 gives the values of the latent roots and vectors for this data set .It also checks the multicollinearity between the variables according to the following conditions:

$$\lambda_j < 1 \qquad and \quad |\gamma_{0j}| < 0.5 \ \ for \quad j = 0,1,2,\ldots,p$$

Table(2): Test results

i	λ_i	γ_{0i}	Conditions
0	0.0003	-0.053	Two holds
1	0.0052	-0.0518	Two holds
2	0.0097	0.8871	One holds
3	0.1122	0.0636	Two holds
4	4.8726	0.4512	One holds

The above table shows that three values satisfy the two conditions. This means that there is a multicollinearity between these variables, so the Latent Root estimators and its variances will depend only on the remaining two values i.e q = 2.

7.4 Table 3 gives the values and the variances of the estimated regression parameters using Ordinary Least Squares equation (2).

Table (3): Estimators, variances and the t- test values of the regression coefficients in the Least Squares

i	$\hat{\beta}_{iOLS}$	$V(\hat{\beta}_{iOLS})$	$t(\hat{\beta}_{iOLS})$
1	-0.2037	0.3087	$-0.3667^{(N.S)}$
2	1.7528	4.4836	$0.8278^{(N.S)}$
3	-0.0104	0.4892	$-0.0148^{(N.S)}$
4	-0.5592	1.4585	$-0.463^{(N.S)}$

We see from the above table that all variables are not significant .Table 4 gives the values and the variances Using Latent Roots method equation (5).

Table (4): Estimators, variances and the t- test values of the regression coefficients in the Latent Root

i	$\hat{\beta}_{iLR}$	$V(\hat{\beta}_{iLR})$	$t(\hat{\beta}_{iLR})$
1	0.196	0.0001386	16.6482
2	0.2168	0.000154	17.4742
3	0.2369	0.0001578	18.8614
4	0.3577	0.0001885	26.0558

We see from the above table that all variables are significant.

7.5 MSE is estimated for the two methods according to equations (4) and (9) respectively is as in table 5:

Table (5): MSE for Ordinary Least Squares and the Latent Roots before deletion

The method	σ	Part1	Part2	MSE	R^2
Least Squares	0.0471	6.7401	0	6.7401	98.89%
Latent Roots	0.0497	0.0006388	0.0028	0.0034	98.76%

Notice that the MSE for Latent Roots method is lower than that for Least Squares method as well as the value of R^2.The MSE and the coefficient of determination values for both methods after ignoring the non-significant variables is given as:

Table (6): MSE for the Least Squares and Latent Roots after deletion

The method	σ	MSE	R^2
Least Squares	There are not significant parameter		
Latent Roots	0.0497	0.0034	98.76%

From the above table we conclude that the estimated model using the Latent Roots method is better than the estimated model in the Least Squares method taking in consideration the number of significant variables for both methods .

7.6 After deleting the variables that are not important in the prediction process, the estimated regression equation in the Latent Roots method is:

$$\hat{y}_i^* = 0.196\, x_1^* + 0.2168\, x_2^* + 0.2369 x_3^* + 0.3577\, x_4^* \ .$$

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ON A CLASS OF HARMONIC FUNCTIONS DEFINED BY A CONVOLUTION DIFFERENTIAL OPERATOR

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ABSTRACT

A class of complex-valued harmonic univalent functions defined by convolution differential operator is introduced. Coefficient bounds, distortion theorem, and other properties of this class are obtained.

Keywords: Harmonic functions; convolution; differentialopertator.

1. INTRODUCTION

In any complex domain G a continuous function $f = u + iv$ is said to be harmonic in G if both u and v are real harmonic in G. In a simply connected domain $D \subset G$ a harmonic complex-valued function might be expressed in term of analytic functions, $h$and g; as $f = h + g$. We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| \geq |g'(z)|$ in D (see[4]). Denote by H the family of functions $f = h + g$, that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = fz(0) - 1 = 0$. Thus for $f = h + g$ in H we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{and} g(z) = \sum_{k=1}^{\infty} b_k z^k \quad 0 \leq b < 1.$$

Note that the family of harmonic univalent functions H, reduces to the class of analytic functions A, which can be written in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

if the co-analytic part of $f = h + g$ is identically zero that is $g \equiv 0$.

In the negative counter part, let T be donate the subclass of H consisting of all functions $f = h + g$where$f$ and g are given by

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \text{and} g(z) = -\sum_{k=1}^{\infty} |b_k| z^k \quad 0 \leq b < 1.$$

See [16].

In [14] Ruscheweyh defined the differential operator

$$R^{\alpha}: A \to A$$

where$\alpha \in \mathbb{N}$ and

$$R^0 f(z) = f(z)$$
$$R^1 f(z) = zf'(z)$$
$$R^2 f(z) = zf'(z) + \frac{1}{2} z^2 f''(z)$$
$$\vdots$$
$$(\alpha + 1) R^{\alpha+1} f(z) = \alpha R^{\alpha} f(z) + z\left(R^{\alpha} f(z)\right)'.$$

If $f\ (z)$ is an analytic function of the form$(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$R^{\alpha} f(z) = z + \sum_{k=2}^{\infty} C(\alpha, k) a_k z^k$$

where $c(\alpha, k) = \binom{k+\alpha-1}{\alpha}$.

In [15] Salagean defined the following differential operator

$$S^n: A \to A$$

where$n \in \mathbb{N}$ and

$$S^0 f(z) = f(z)$$
$$S^1 f(z) = zf'(z)$$
$$\vdots$$
$$S^n f(z) = z\left(S^{n-1} f(z)\right)'.$$

If $f\ (z)$ is an analytic function of the form $(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

Later Al-Oboudi [1] introduced a generalisation of Salagean operator which defined as follows:

$$D_{\lambda}{}^{n}: A \to A$$

where$n \in \mathbb{N}_0$ and

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = (\lambda + 1) f(z) + \lambda z f'(z) = D_{\lambda} f(z)$$
$$\vdots$$
$$D^n f(z) = D_{\lambda}\left(D^{n-1} f(z)\right) \ .$$

If $f\ (z)$ is an analytic function of the form $(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k.$$

In [5] Darus and Al-Shaqsi introduced the differential operator

$$R_{\alpha,\lambda}{}^{n}: A \to A$$

where$n \in \mathbb{N}$ and

$$R_{\alpha,\lambda}{}^{0} f(z) = f(z)$$
$$R_{\alpha,\lambda}^{n} f(z) = zf'^{(z)} + \lambda z^2 f''(z) = R^* f(z)$$
$$\vdots$$
$$R_{\alpha,\lambda}{}^{n} f(z) = R^* \left(R_{\alpha,\lambda}{}^{n-1} f(z)\right) \ .$$

If $f\ (z)$ is an analytic function of the form $(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$R_{\alpha,\lambda}^{n} f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n C(\alpha, k) a_k z^k.$$

In [10] Lupas considered the differential operator SR^nwhich is the convolution of $R^{\alpha}$and S^n.More precisely,

$$SR^n\ f(z) = R^{\alpha} f(z) * S^n f(z)$$

$$= \left(z + \sum_{k=2}^{\infty} C(\alpha, k) a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} k^n a_k z^k\right)$$
$$= z + \sum_{k=2}^{\infty} C(\alpha, k) k^n a^2{}_k z^k.$$

In [2] Andrei considered the differential operator DR^nwhich is the convolution of $D^n$and R^α. More precisely,

$$DR^n f(z) = R^\alpha f(z) * D^n f(z)$$
$$= \left(z + \sum_{k=2}^{\infty} C(\alpha, k) a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k\right)$$
$$= z + \sum_{k=2}^{\infty} C(\alpha, k)[1 + \lambda(k-1)]^n a^2{}_k z^k.$$

To this end, the platform is ready to construct new convoluted differential operator. Let usconsider the differential operators $D^n$and ${R_{\alpha,\lambda}}^n$. Then, the convoluted operator of both of them is

$$\widetilde{D}^n f(z) = D^n f(z) * R^n_{\alpha,\lambda} f(z)$$
$$= \left(z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n C(\alpha, k) a_k z^k\right)$$
$$= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^{2n} C(\alpha, k) a_k^2 z^k.$$

In 2002 Jahangiri et al. [7] introduced the modified Salagean operator of harmonic univalent function. In 2003; Murugusundaramoorthy [13] introduced the modified Ruscheweyh of harmonic univalent function. In the next definition we will modify the operator DR^n to harmonic univalent function.

Definition 1.1. For harmonic function $f = h + g$, we define the following differential operator

$$\widetilde{D}^n f(z) = \widetilde{D}^n h(z) + \overline{\widetilde{D}^n g(z)}$$

where , $n \in \mathbb{N}$.

Recently, many researchers have showed an interest to invent classes of harmonic functions defined by differential operators, convolution, and subordination. See [3], [6], [8], and [9].

We let $D_H(n, \alpha, \lambda, \mu)$ denote the family of harmonic functions $f = h + g$ for which

$$Re\left\{\left(\widetilde{D}^n f(z)\right)'\right\} \geq \mu.$$

We further denote by $D_T(n, \alpha, \lambda, \mu)$, the subclass of $D_H(n, \alpha, \lambda, \mu)$ where

$$D_T(n, \alpha, \lambda, \mu) = T \cap D_H(n, \alpha, \lambda, \mu).$$

2. COEFFICIENT BOUNDS

In this section, coefficient bounds of the classes $D_H(n, \alpha, \lambda, \mu)$and $D_T(n, \alpha, \lambda, \mu)$ are given.

Theorem 2.1. Let $f = h + \bar{g}$ be harmonic function, $0 \leq \mu < 1, n, \alpha \in \mathbb{N}_0, a_1 = 1, \lambda \geq 0$. If

$$\sum_{k=2}^{\infty} \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|a_k| + \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|b_k| \le 2$$

where

$$\varphi(n,k,\lambda,\alpha) = [1+\lambda(k-1)]^{2n} C(\alpha,k).$$

Then f is sense preserving, harmonic univalent in U and $f \in D_H(n,\alpha,\lambda,\mu)$.

*Proof.*Note first that

$$|h'(z)| \ge 1 - \sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|a_k||z^{k-1}|$$
$$> \sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|b_k||z^{k-1}|$$
$$\ge |g'(z)|,$$

so that f is locally univalent and sense preserving.

To show that f is univalent in U, we consider that the restriction in the theorem hold. If $g(z) = 0$, then f is analytic. And then, the univalenceof f comes from its close-to convexity. If $g(z) \neq 0$and z_1, z_2 are any distinct points in U, then

$$\frac{|f(z_1)-f(z_2)|}{|h(z_1)-h(z_2)|} > 1 - \frac{|g(z_1)-g(z_2)|}{|h(z_1)-h(z_2)|}$$
$$= 1 - \left|\sum_{k=2}^{\infty} \frac{b_k(z_1^k - z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right|$$
$$> 1 - \frac{\sum_{k=2}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|}$$
$$\ge 1 - \frac{\sum_{k=2}^{\infty} \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|b_k|}{1-\sum_{k=2}^{\infty} \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|a_k|}$$
$$\ge 0.$$

Therefore, f is univalent.

Using the fact that $Re\ w \ge \mu$if and only if $|1-\mu+w| \ge |1+\mu-w|$it suffices to show that

$$\left|1-\mu+\left(\widetilde{D}^n f(\mathrm{z})\right)'\right| \ge \left|1+\mu-\left(\widetilde{D}^n f(\mathrm{z})\right)'\right|.$$

To do so, we have

$$\left|1-\mu+\left(\widetilde{D}^n f(\mathrm{z})\right)'\right| \ge \left|1+\mu-\left(\widetilde{D}^n f(\mathrm{z})\right)'\right|$$

$$\ge 2(1-\mu) - 2\sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|a_k||z|^{k-1} - 2\sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|b_k||z|^{k-1}$$
$$> 2(1-\mu) - \left\{1 - \left(\sum_{k=2}^{\infty} \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|a_k| + \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|b_k|\right)\right\}$$

which is nonnegative by the theorem restriction, and so $f \in D_H(n,\alpha,\lambda,\mu)$▷□

Next theorem provides a coefficient bounds for the class $D_T(n,\alpha,\lambda,\mu)$.

Theorem 2.2. Let $f = h + \bar{g}$ be harmonic function. Then $f \in D_T(n, \alpha, \lambda, \mu)$ if and only if

$$\sum_{k=2}^{\infty} \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|a_k| + \frac{\varphi(n,k,\lambda,\alpha)}{1-\mu}|b_k| \leq 2.$$

*Proof.*Since $D_T(n, \alpha, \lambda, \mu) \subset D_H(n, \alpha, \lambda, \mu)$we only need to prove the (only if) part of the theorem. To do so, assume that $f \in D_H(n, \alpha, \lambda, \mu)$▹Then by the assertion we have

$$Re\left\{\left(\widetilde{D}^n f(\mathrm{z})\right)'\right\} = Re\left\{\left(\widetilde{D}^n h(\mathrm{z})\right)' + \overline{\left(\widetilde{D}^n g(\mathrm{z})\right)'}\right\}$$

$$= Re\left\{1 - \sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|a_k|z^{k-1} - \sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|b_k|\overline{z^{k-1}}\right\} > \mu$$

If we choose $z$to be real and let $z \to 1^{-}$we get

$$1 - \sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|a_k| - \sum_{k=2}^{\infty} \varphi(n,k,\lambda,\alpha)|b_k| > \mu.$$

Which is precisely the assertion of Theorem 2.2.□

3. DISTORTION THEOREM AND EXTREME POINTS

In this section, distortion theorem of the class $D_T(n, \alpha, \lambda, \mu)$is obtained.

Theorem 3.1. If $f \in D_T(n, \alpha, \lambda, \mu)$, $0 \leq \mu < 1, n, \alpha \in \mathbb{N}_0, a_1 = 1; \lambda \geq 0$, and $|\mathrm{z}| = r < 1$, then

$$|f(z)| \leq 1 + |b_1|r\left(\frac{1-\mu}{\varphi(n,2,\lambda,\alpha)} - \frac{\varphi(n,1,\lambda,\alpha)}{\varphi(n,2,\lambda,\alpha)}|b_1|\right)r^2$$

and

$$|f(z)| \geq 1 - |b_1|r\left(\frac{1-\mu}{\varphi(n,2,\lambda,\alpha)} - \frac{\varphi(n,1,\lambda,\alpha)}{\varphi(n,2,\lambda,\alpha)}|b_1|\right)r^2$$

*Proof.*The proof follows,immedeality, by the coefficient bound of the class $D_T(n, \alpha, \lambda, \mu)$.□

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STOCHASTIC DELAY DIFFERENTIAL EQUATIONS OF PREY PREDATOR SYSTEM WITH HUNTING COOPERATION: ANALYRIC AND NUMERIC

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ABSTRACT

In this paper, we investigate the dynamics of a stochastic delay differential equations (SDDEs) of predator-prey system with hunting cooperation on predator. To prove the existence of global positive solution, we use Milstein's scheme, to solve SDDEs of the prey-predator system. Sufficient criteria for global existence are obtained. The increase of the noise intensity has a drastic impact on the dynamical behavior of both species with or without the delay effect. Time-delay plays a vital role in population dynamics of prey-predator, which has been recognized to contribute critically to the stable or unstable outcomes of prey population due to predation. Illustrative numerical examples are provided to show the effectiveness of the theoretical results.

Keywords: Hunting cooperation; Milstein's scheme; Stochastic Prey-Predator model; Time-delay

1. INTRODUCTION

The study of prey-predator systems between two or more species to model life system interactions is an important issue in biological systems (see, e.g., [5, 6, 10]). The dynamical relationship between predator and their preys has been essential in theoretical ecology since the famous Lotka-Volterra equations [7, 13], which is a pair of first order, nonlinear differential equations that describes the dynamics of biological systems in which two species interact. The system parameters have main role to determine the qualitative properties of predator prey systems.

One major component of the predator-prey relationships is functional response, which is refer to the change in the density of prey attached per unit time per predator as the prey density changes. In [3], Holling discussed three different types of functional response to model the phenomena of predation, the Holling type-I is of the form $p(x) = nx$ and the Holling type-II is of the form $p(x) = nx/(b + x)$, where x is the population density, n is the maximum rate of predation, and b is the half saturation constant. Predator hunting cooperation can be considered in the formulation of functional response, depends on prey and predator densities. Consuming rate by predator increases as predator density increase. Thus, when the prey density become low, hunting cooperation can be adverse to predator population itself.

Time-delays (time-lags) are incorporated into biological systems to represent the time requiredfor maturation period, reaction time, feeding time, etc. See [9]. Herein, we incorporate time-delay in the model for the gestation period of preys. It is also interesting to study the impact of hunting cooperation in the dynamical complexities for the underlying model.
Systems are often subject to environmental noise, which is important factor in ecosystems, to suppress a potential population explosion. In reality, natural phenomena counter an environmental noise and usually do not follow strictly deterministic laws but oscillate randomly about some average values, so that the population density never attains a fixed value with the advancement of time [2, 11]. Furthermore, environmental stochasticity can affect large populations, as well as small. In [1] the authors studied the effect of environmental fluctuations on acompetitive model for two phytoplankton species where one species liberate toxic substances by considering a discrete time delay parameter in the growth equations of

both species. Recently, some authors include the environmental noise into deterministic biological models to show the stochastic perturbation effects.

In this paper, we deal with stochastic delayed prey predator model with hunting cooperation on predator. In Section 2, we formulate a SDDEs prey predator model then we discuss the qualitative behavior of the deterministic model and study the existence and uniqueness of global positive solutions. Some numerical simulations are obtained in Section 3. Section 4 contains conclusions.

2. MODEL FORMULATION AND MAIN RESULTS

Consider the following prototypical delayed predator-prey model considering intra-specific com- petition among predator and delay logistic growth functions for the prey

$$\begin{aligned} \frac{dx(t)}{dt} &= r\,x(t)\left(1-\frac{x(t-\tau_1)}{K}\right)-f\big(x(t),y(t)\big)y(t) \\ \frac{dy(t)}{dt} &= \mu f\big(x(t-\tau_2),y(t)\big)y(t)-\delta y(t)-ay^2(t) \end{aligned} \tag{1}$$

with initial conditions

$$\begin{aligned} x(\theta)&=\varphi_1(\theta)>0,\theta\in[-\tau_1,0),\varphi_1(0)>0, \\ y(\theta)&=\varphi_2(\theta)>0,\theta\in[-\tau_2,0),\varphi_2(0)>0 \end{aligned} \tag{2}$$

where x(t) and y(t) stands for population densities of prey and predator. The time delays τ_1, τ_2 isincorporatedtoconsiderthegestationtime,$\varphi_1$and$\varphi_2$arecontinuousboundedfunctionsin the intervals $[\tau_1, 0]$ and $[\tau_2, 0]$ respectively. The intrinsic growth rate of prey is denoted by *r*, where *K* is the environmental carrying capacity, δ is the death rate for predator, *a* is the predatorintra-specificcompetitionrate.Functionalresponse$f(x,y)$dependonbothpredator andpreydensitiesandμistheconversionefficiency($0<\mu<1$).AssumethattypeIIfunctional response has the form $\sigma x/(1 + c\sigma x)$, where $\sigma$is the consumption rate of prey by their predator and *c* is the handling time of the predator. We presume consumption rate depending on the predatordensitytoinducepredatorcooperationforhuntingtheprey.Therefore,wetake$\alpha>0$ is the cooperativehunting parameter.Hence,the functionalresponse takes theform$f(x,y)=(1+\alpha y)x/(1+c(1+\alpha y)x)$.

Herein,wewillstudytheeffectoffluctuatingenvironmentonthedynamicbehaviorof(1),with time delays (τ_1&τ_2) which are introduced in the growth components for each of the species. In ordertostudytheeffectofenvironmentaldrivingforceonthedynamicbehaviorofthedelayed model we incorporate white noise terms into the growth equations of both prey and predator, thencorrespondingtosystem(1)weobtainedthefollowingstochasticdelayedmodel

$$\begin{aligned} dx(t) &= \left[r\,x(t)\left(1-\frac{x(t-\tau_1)}{K}\right)-f\big(x(t),y(t)\big)y(t)\right]dt+\sigma_1 x(t)dB_1(t) \\ dy(t) &= \left[\mu f\big(x(t-\tau_2),y(t)\big)y(t)-\delta y(t)-ay^2(t)\right]dt+\sigma_2 y(t)dB_2(t) \end{aligned} \tag{3}$$

with initial conditions (2), by assuming $\theta\in[-\tau, 0]$, $\tau=\max\{\tau_1, \tau_2\}$, i.e. $(x_0, y_0)=(\varphi_1, \varphi_2)^T\in C([-\tau,0],R_+^2)$ *with* $R_+^2=\{(x,y)\in R^2: x>0, y>0\}$, *if* $x\in R^2$, *its norm is denoted by*$|x|=\sqrt{x_1^2+x_2^2}$.$B_1(t), B_2(t)$ *are standard independent Wiener processes defined on a complete probability space* $(\Omega, A, \{A\}_{t\geq 0}, P)$ *with a filtration* $\{A\}_{t\geq 0}$ *satisfying the usual conditions; and* σ_1, σ_2 *are the positive intensities of white noises.*

3. QUALITATIVE BEHAVIOUR OF THE DETERMINISTIC MODEL

Before analyzing the dynamics of model (3), we discuss the following results for the delayed model (1) with initial conditions (2), for simplicity we consider K = 1, then the Jcobian matrix at the interior equilibrium $E^*(x^*, y^*)$ is given by

$$J = \begin{bmatrix} A_1 + B_1 e^{-\lambda\tau_1} & A_2 \\ B_2 e^{-\lambda\tau_2} & A_3 \end{bmatrix}$$

[redacted]

$$A_1 = 1 - x^* - \frac{(1+\alpha y^*)}{(1+c(1+\alpha y^*)x^*)^2}, A_2 = \frac{-x^*(1+\alpha y^*)}{(1+c(1+\alpha y^*)x^*)},$$
$$A_3 = \frac{\mu(1+\alpha y^*)x^*}{(1+c(1+\alpha y^*)x^*)} - \delta - 2ay^*, \qquad B_1 = -x^*, B_2 = \frac{\mu(1+\alpha y^*)y^*}{(1+c(1+\alpha y^*)x^*)}$$

[redacted]

$$\lambda^2 - (A_1 + A_3)\lambda + A_1A_3 + (B_1A_3 - B_1\lambda)e^{-\lambda\tau_1} - A_2B_2e^{-\lambda\tau_2} = 0, (4)$$

[redacted]

- $\tau_1 = \tau_2 = 0$ [redacted]
- (ii) $\tau_1 > 0, \tau_2 = 0$ (iii) $\tau_2 > 0, \tau_1 = 0$ [redacted]
- $\tau_1 = \tau_2 > 0$ [redacted]

25.

[redacted]

$$-w^2 - (A_1 + A_2)(wi) + A_1A_3 - A_2B_2 + \big(B_1A_3 - B_1(wi)\big)e^{-w\tau_1} = 0 \qquad (5)$$

[redacted]

$$w^4 + q_1w^2 + q_2 = 0. \qquad (6)$$

where $q_1 = A_1^2 + A_3^2 + 2A_2B_2 - B_1^2$, and $q_2 = (A_1A_3 - A_2B_2)^2 - B_1^2A_3^2$. The local stability of E^* depends on the values of q1 & q2. Therefore, equation (6) has positive root if $q_1>0$ and $q_2<0$, therefore, it has a pair of pure imaginary roots of the form iw_0, then from (5) we get

$$\tau_{ij} = \frac{1}{w_0}[arccos\left[\frac{(A_1A_3 - A_2B_2 - w_0^2)B_1A_3}{B_1^2w_0^2 + B_1^2A_3^2} + \frac{B_1w_0^2(A_1+A_3)}{B_1^2w_0^2 + B_1^2A_3^2}\right] + 2j\pi (7)$$

where j = 0, 1, 2,. .. , we arrive at the following theorem:

Theorem 1 The interior equilibrium point E^* will be stable for $\tau < \tau_1^*$, where τ_1^* is obtained from (7) by taking j =0 from.. For $\tau > \tau_1^*$, E^* will be unstable, and for $\tau = \tau_1^*$ it has a periodic solution.

Now we study the existence of Hopf bifurcation with respect to the bifurcation parameter τ_1.

Theorem 2 System (1) undergoes Hopf bifurcation at the interior equilibrium E^* when $\tau_1 = \tau_{1j}$

where τ_{1j} is given by (7), such that $R(\frac{d\lambda}{d\tau})^{-1} > 0$.

Proof: We differentiate (5) with respect to τ_1, then substitute $\lambda = iw_0$ after simplifying we obtain

$$R(\frac{d\lambda}{d\tau})^{-1} = \frac{2\left(w_0^4-(A_1A_3-A_2B_2)w_0^2\right)+(A_1+A_3^2)w_0^2}{(A_1+A_3)^2w_0^4+(w_0^3-(A_1A_3-A_2B_2))^2} - \frac{B_1^2w_0^2}{B_1^2w_0^4+B_1^2A_3^2w_0^2} \quad (8)$$

25.1. *Existence and uniqueness of the global positive solution*

Herein, we will show the positivity of solution for model (3) and we will prove that the enviromental noise holds the explosion for the delay equation.

Theorem-3 Let $\boldsymbol{Z(t)} = (\boldsymbol{x(t)}, \boldsymbol{y(t)})$ and $|\boldsymbol{Z(t)}| = \sqrt{\boldsymbol{x^2(t)} + \boldsymbol{y^2(t)}}$, then for any given initial value $Z(\kappa) = \{(x(\kappa), y(\kappa)): -\tau\leq\kappa\leq 0\}\in C([-\tau, 0]; R_+^2)$ there exist a unique positive solution $Z(t)$
To (3) on $t\geq -\tau$ and the solution will remain in R_+^2 with property one.

Proof: From the biological point view, we will take into our consideration the positive solution to model (3), assuming that $x(t) = e^{n(t)}$, $y(t) = e^{p(t)}$ and applying Ito^'s formula model an be reformulated as follows

$$dn(t) = \left(1 - e^{n(t-\tau_1)} - \frac{(1 + \alpha e^{p(t)})e^{p(t)}}{1 + c(1 + \alpha e^{p(t)})e^{n(t)}} - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t)$$

$$dp(t) = \left(\frac{\mu(1+\alpha e^{p(t)})e^{n(t-\tau_2)}}{1+c(1+\alpha e^{p(t)})e^{n(t-\tau_2)}} - \delta - \frac{\sigma_2^2}{2}\right)dt + \sigma_2 dB_2(t). \quad (9)$$

All the of (9) satisfy the local Lipschitz condition, then for any initial values $n(\kappa) = \ln x(\kappa)$, $p(\kappa) = \ln y(\kappa)$, $\kappa\in[-\tau, 0]$, there is a unique local solution n(t), p(t) on $[-\tau, t_e)$, where t_e is the explosion time. In order to show that the solution is global, it is sufficient to show $t_e = \propto$ a.s. Let $l_0 > 0$ be sufficiently large so that $Z(t) = \left\{\left(\phi_1(t), \phi_2(t)\right): -\tau\leq t\leq 0\right\} \in C([-\tau, 0]; R_+^2)$ all lie within the interval $[\frac{1}{l_0}, l_0]$. For each integer $l\geq l_0$, define the stopping time

$$t_l = \inf\{t\in[0, t_e) : x(t)\notin\left(\frac{1}{l}, l\right), y(t)\notin\left(\frac{1}{l}, l\right),$$

where we set $\inf\Phi = \propto$. We consider t_l is increasing as $l\to\propto$. Let $t_\propto = lim_{t\to\propto} t_l$, then $t_\propto\leq t_e$ a.s. If we show $t_\propto = \propto$ a.s. and $Z(t)\in R_+^2$ a.s. for all $t\geq 0$. To show this statement, we define a C^2- function $V: R_+^2\to R_+$ by $V(Z) = V_1 + V_2$.
Where $V_1 = (x - \log x - 1) + (y - \log y - 1)$, and $V_2 = \int_t^{t+\tau}[x^2(s-t) + x(s-t)]ds$. It is easy to see that function V(Z) is non-negative. The rest of the proof follows that of [8].

4. NUMERICAL SIMULATIONS

In this section, we carry out some numerical simulations to display the qualitative behaviors of model (3) for different values of τ and σ_1, σ_2, note that model (3) has multiplicative noise. We utilize Milstein's scheme [4] to illustrate our findings. In Fig. 1 we simulate model (3) when $\sigma_1 = \sigma_2 = 0$ such that $\tau = 0.2$ &$\tau = 0.8$ as in (a) & (c) respectively, and we observe that the solution is asymptotic stable as in (a), if we increase the value of the environmental noise $\sigma_1 = \sigma_2 = 0.001$ and keeping $\tau = 0.2$ we can find a stochastically stable solution (b). Periodic solution as in (c) is shown. Thus, by increasing the environmental noise $\sigma_1 = \sigma_2 = 0.001$ with the same magnitude of time delay the amplitude of stochastic fluctuation increases significantly as in (d).

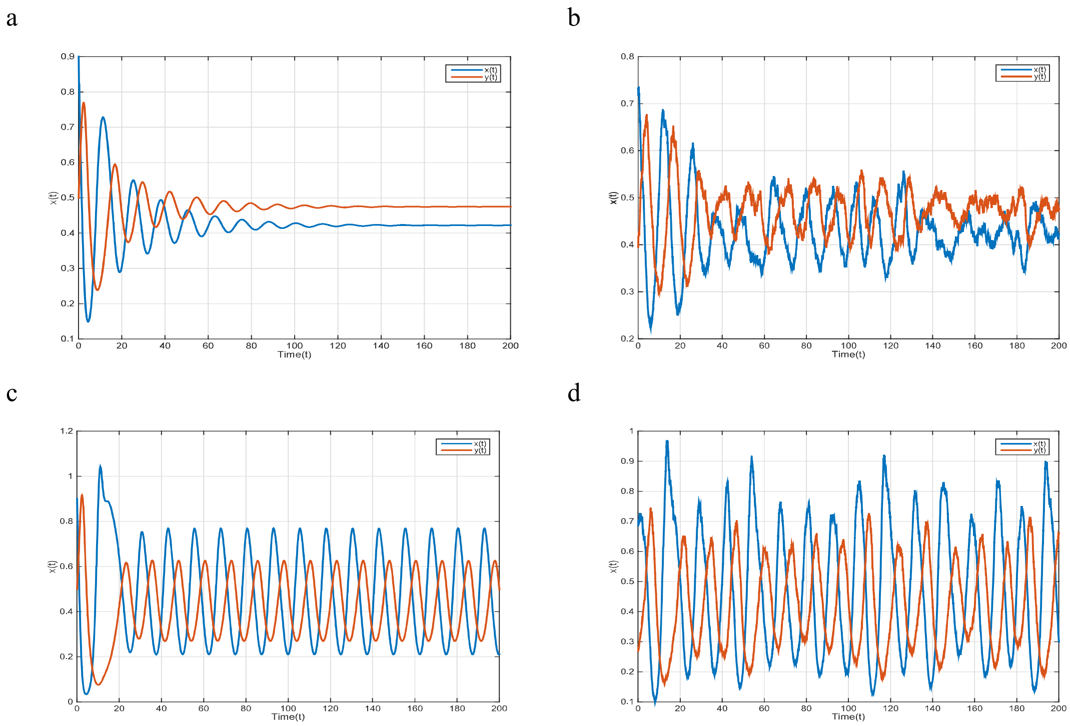

Figure1: Numerical simulations of the solution of the stochastic model (3) with parameter values α=1.6, a=0.05, c=0.6, δ=0.49, K=1, and μ=0.9. (a) when $\sigma_1 = \sigma_2 = 0$ and τ =0.2, (b) when$\sigma_1 = \sigma_2 = 0.001$ and τ =0.2 which shows stochastically stable population distribution for both species. Periodic solution for $\sigma_1 = \sigma_2 = 0$ and τ =0.8 as in (c), while in (d) when $\sigma_1 = \sigma_2 = 0.001$ and τ =0.8

CONCLUSION

In this work, we studied a stochastic predator-prey system with time-delay and hunting cooperation on predator. We defined the characteristic equation of the deterministic model. Some new and interesting sufficient conditions that ensure the local asymptotic stability for the addressed model have been derived. We attained critical value of time delay where Hopf bifurcation occurs. Existence and uniqueness of the positive global solution for such SDDEs model have been discussed. The theoretical results and numerical simulations of SDDEs model, we have seen thatfor small values of white noise has a significant impact on the dynamical behavior of the model. The combination of stochastic effects and time delay increases the complexity and enriches the dynamics of the model.

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LOCAL EXISTENCE AND BLOW UP OF SOLUTIONS FOR A TIME-SPACE FRACTIONAL EVOLUTION SYSTEM WITH NONLINEAR TIME-NONLOCAL SOURCE TERMS

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ABSTRACT

In this paper, we are concerned with local existence and blow-up of a unique solution to the Cauchy problem for a time-space fractional evolution system with time-nonlocal source terms of polynomial growth. At first, we prove the existence and uniqueness of the local mild solution by the Banach contraction mapping principle. Then, we show that such a mild solution is a weak solution and we establish a blow-up result by the test function method with a judicious choice of the test function. Finally, we establish an estimate of the life span of blowing up solutions under some suitable conditions.

Keywords: Fractional derivatives and integrals; nonlinear evolution equations; local existence; blow-up; life span

1. INTRODUCTION

In this paper, we consider the following Cauchy problem

$$\begin{cases} \boldsymbol{D}_{0|t}^{\alpha_1} u + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha_1}(|v|^{p-1}v), \ x \in IR^N, t > 0, \\ \boldsymbol{D}_{0|t}^{\alpha_2} v + (-\Delta)^{\beta/2} v = J_{0|t}^{1-\alpha_2}(|u|^{q-1}u), \ x \in IR^N, t > 0, \\ \quad u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ x \in IR^N, \end{cases} \tag{1}$$

where $N \geq 1,\ 0 < \alpha_1, \alpha_2 < 1,\ 0 < \beta \leq 2,\ \boldsymbol{D}_{0|t}^{\alpha_i}$is the Caputo fractional derivative operator of order α_i, $J_{0|t}^{1-\alpha_i}$ is the left-sided Riemann-Liouville fractional integral of order $1-\alpha_i$, defined by

$$J_{0|t}^{1-\alpha_i} f(t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-s)^{-\alpha_i} f(s) ds,$$

where Γ is the gamma function, $(-\Delta)^{\beta/2}$ is the fractional Laplacian operator defined by

$$(-\Delta)^{\beta/2} w(x) = \mathcal{F}^{-1}\big(|\xi|^{\beta} \mathcal{F}(w)(\xi)\big)(x),$$

for $w \in D\big((-\Delta)^{\beta/2}\big) = H^{\beta}(IR^N)$, where $H^{\beta}(IR^N)$ is the homogeneous Sobolev spacedefined by

$$H^{\beta}(IR^N) = \{w \in S';\ (-\Delta)^{\beta/2} w(x) \in L^2(IR^N)\}, \text{ if } \beta \notin IN,$$
$$H^{\beta}(IR^N) = \{w \in L^2(IR^N);\ (-\Delta)^{\beta/2} w(x) \in L^2(IR^N)\}, \text{ if } \beta \in IN,$$

where S' is the Schwartz space, $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ its inverse, and $u_0, v_0 \in C_0(IR^N)$, where $C_0(IR^N)$ denotes the space of all continuous functions decaying to zero at infinity.

If $\boldsymbol{D}_{0|t}^{\alpha_i}$ is replaced by the first differential operator d/dt we have the following problem studied by Fino and Kirane [4],

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$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha_1}(|v|^{p-1}v), x \in IR^N, t > 0, \\ v_t + (-\Delta)^{\beta/2} v = J_{0|t}^{1-\alpha_2}(|u|^{q-1}u), x \in IR^N, t > 0, \\ \quad u(x,0) = u_0(x), v(x,0) = v_0(x), x \in IR^N. \end{cases}$$

First, they studied the case $\beta = 2$. They validated the system by an existence-uniqueness result. And then they gave the blow-up rate of solutions and the necessary conditions for local or global existence. Finally, in [4, Remark 2], they claimed that using the same method, one can extend this case to $0 < \beta < 2$.
This paper is organized as follows: In section 2, we present some definitions and results that will be used throughout this study. In section 3, the local existence and uniqueness of mild solutions of problem (1)are established. In Section 4, blowing-up solutions are shown to exist, while in Section 5, we establish an estimate of the life span of blowing up solutions.

2. PRELIMINARIES

In this section, we present some definitions and results that will be used in the following sections, which can be found in [2, 5]. Let α be a real constant such that $0 < \alpha < 1$.
The Caputo derivative of order α, for a differentiable function f, is defined by

$$\boldsymbol{D}_{0|t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds.$$

The left-sided Riemann-Liouville fractional derivative of order α, for a continuous function f, is defined by

$$D_{0|t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds.$$

The right-sided Riemann-Liouville fractional derivative of order α, for a continuous function f, is defined by

$$D_{t|T}^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (t-s)^{-\alpha} f(s) ds.$$

Furthermore, for every $f, g \in C([0,T])$ such that $D_{0|t}^{\alpha} f$, $D_{t|T}^{\alpha} g$ exist and are continuous, the formula of integration by parts can be given by

$$\int_0^T g(t) D_{0|t}^{\alpha} f(t) dt = \int_0^T f(t) D_{t|T}^{\alpha} g(t) dt.$$

The relation between Caputo and Riemann-Liouville derivatives is

$$\boldsymbol{D}_{0|t}^{\alpha} f(t) = D_{0|t}^{\alpha}[f(t) - f(0)].$$

The Mainardi's function is given by

$$M_\alpha(z) = \sum_{k=1}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \ 0 < \alpha < 1, \ z \in \mathbb{C}.$$

The Mittag-Leffler operators based on the analytic semigroup $T(t)$ generated by the space fractional operator $(-\Delta)^{\beta/2}$ are defined by

$$P_{\alpha,\beta}(t) = \int_0^\infty M_\alpha(s) T(st^\alpha) ds = \int_0^\infty M_\alpha(s) e^{-st^\alpha(-\Delta)^{\beta/2}} ds,$$

$$S_{\alpha,\beta}(t) = \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha) ds = \int_0^\infty \alpha s M_\alpha(s) e^{-st^\alpha(-\Delta)^{\beta/2}} ds.$$

3. LOCAL EXISTENCE

In this section, we give the local existence and uniqueness of mild solution of the problem (1). First, we give the definition of mild solution of (1).

Definition3.1. (Mild solution) Let $u_0, v_0 \in C_0(IR^N)$ and $T > 0$.We say that $(u,v) \in C([0,T], C_0(IR^N)) \times C([0,T], C_0(IR^N))$ is a mild solution of (1) if (u,v) satisfies, for $t \in [0,T]$, the following equations

$$u(t) = P_{\alpha_1,\beta}(t)u_0 + \int_0^t (t-s)^{\alpha_1-1} S_{\alpha_1,\beta}(t-s) J_{0|s}^{1-\alpha_1}(|v|^{p-1}v)ds,$$
$$v(t) = P_{\alpha_2,\beta}(t)v_0 + \int_0^t (t-s)^{\alpha_2-1} S_{\alpha_2,\beta}(t-s) J_{0|s}^{1-\alpha_2}(|u|^{q-1}u)ds.$$

Theorem 3.2. *(Local existence) Let $u_0, v_0 \in C_0(IR^N)$. Then, there exists a maximal time $T_{max} > 0$ such that the problem* (1) *has a unique mild solution $(u,v) \in C([0,T_{max}), C_0(IR^N)) \times C([0,T_{max}), C_0(IR^N))$. Furthermore,*

$$either\ T_{max} = +\infty\ or\ T_{max} < +\infty\ with \lim_{t \to T_{max}} \left(\|u(t)\|_{L^\infty(IR^N)} + \|v(t)\|_{L^\infty(IR^N)} \right) = +\infty.$$

If, in addition, $u_0, v_0 \geq 0, u_0 \not\equiv 0, v_0 \not\equiv 0$, then $u(t), v(t) > 0$ for all $0 < t < T_{max}$.

4. BLOWING UP SOLUTIONS

Definition 4.1. (Weak solution). Let $u_0, v_0 \in L^\infty_{loc}(IR^N)$ and $T > 0$. We say that (u,v) is a weak solution of (1) if $(u,v) \in L^p((0,T), L^\infty_{loc}(IR^N)) \times L^p((0,T), L^\infty_{loc}(IR^N))$ and satisfies the following equations

$$\int_0^T \int_{\mathbb{R}^N} u_0 \mathbf{D}_{t|T}^{\alpha_1} \psi_1(x,t) dxdt + \int_0^T \int_{\mathbb{R}^N} J_{0|t}^{1-\alpha_1}(|\ v\ |^{p-1}\ v) \psi_1(x,t) dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} u(x,t)(-\Delta)^{\beta/2} \psi_1(x,t) dxdt + \int_0^T \int_{\mathbb{R}^N} u(x,t) \mathbf{D}_{t|T}^{\alpha_1} \psi_1(x,t) dxdt,$$

$$\int_0^T \int_{\mathbb{R}^N} v_0 \mathbf{D}_{t|T}^{\alpha_2} \psi_2(x,t) dxdt + \int_0^T \int_{\mathbb{R}^N} J_{0|t}^{1-\alpha_2}(|\ u\ |^{q-1}\ u) \psi_2(x,t) dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} v(x,t)(-\Delta)^{\beta/2} \psi_2(x,t) dxdt + \int_0^T \int_{\mathbb{R}^N} v(x,t) \mathbf{D}_{t|T}^{\alpha_2} \psi_2(x,t) dxdt,$$

for all test functions $\psi_1, \psi_2 \in C^1([0,T], H^\beta(\mathrm{IR^N}))$ such that $\psi_1(x,T) = \psi_2(x,T) = 0$.

Lemma 4.2.[3] *Let $u_0, v_0 \in C_0(IR^N), T > 0$ and $(u,v) \in C([0,T], C_0(IR^N))^2$ be a mild solution of* (1)*. Then (u,v) is also a weak solution of* (1).

Theorem 4.3. *Let $u_0, v_0 \in C_0(IR^N)$ with $u_0 \geq 0$, $v_0 \geq 0, u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. If*

$$\frac{N}{\beta} < \min\left\{ \frac{1}{\alpha_2(p-1)}, \frac{1}{\alpha_1(q-1)} \right\},$$

then the solution of (1) *blows-up in a finite time.*

5. LIFE SPAN OF BLOWING UP SOLUTIONS

In this section, we give an upper bound estimate of the life span of the blowing up solutions with some special initial datum. To this aim, we consider the following problem

$$\begin{cases} \boldsymbol{D}_{0|t}^{\alpha_1} u_\varepsilon + (-\Delta)^{\beta/2} u_\varepsilon = J_{0|t}^{1-\alpha_1}(|v_\varepsilon|^{p-1} v_\varepsilon),\ x \in IR^N, t > 0, \\ \boldsymbol{D}_{0|t}^{\alpha_2} v_\varepsilon + (-\Delta)^{\beta/2} v_\varepsilon = J_{0|t}^{1-\alpha_2}(|u_\varepsilon|^{q-1} u_\varepsilon),\ x \in IR^N, t > 0, \\ \quad u_\varepsilon(x,0) = \varepsilon u_0(x), v_\varepsilon(x,0) = \varepsilon v_0(x),\ x \in IR^N, \end{cases} \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $0 < \alpha_1, \alpha_2 < 1$, $0 < \beta \leq 2$ and $u_0, v_0 \in C_0(IR^N)$ satisfies

$$u_0(x) \geq m_0|x|^{-\delta}, v_0(x) \geq n_0|x|^{-\delta}, |x| \geq \varepsilon_0, N < \delta < \frac{\tilde{\delta}}{\alpha}, \tag{3}$$

for some positive constants m_0, n_0 and ε_0, where

$$\alpha = \max\{\alpha_1, \alpha_2\},\ \widetilde{\delta} = \min\left\{(\alpha - \alpha_1)N + \frac{\beta}{p-1}, (\alpha - \alpha_2)N + \frac{\beta}{q-1}\right\}.$$

Theorem 5.1. *Suppose that (3) holds. Let* $[0, T_{max})$ *be the life span of the solution* $(u_\varepsilon, v_\varepsilon)$ *of the problem (2). Then there exists a positive constant* C *such that*

$$T_\epsilon \leq C\epsilon^{\frac{1}{\eta}}, \eta = \frac{\alpha\delta}{\beta} - \frac{\alpha N}{\beta} + \max\left\{\frac{\alpha_1 N}{\beta} - \frac{1}{q-1}, \frac{\alpha_2 N}{\beta} - \frac{1}{p-1}\right\} < 0.$$

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SOLVING FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF ORDER 2β USING FRACTIONAL POWER SERIES METHOD

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ABSTRACT

The solution of fractional integro-differential equations, in the Volterra sense, is very important to describe the behavior of linear and non-linear problems. In this article, we discuss the analytical approximate solution for a class of fractional Volterra integro-differential equations of order 2β, where $0 < \beta \leq 1$. The fractional power series method (FPSM) is applied to provide the analytical solutions in the form of rapidly convergent fractional power series (FPS) depending on the residual error function and Taylor series generalized formula under the Caputo sense. In order to validate the effectiveness, potential, and simplicity of the proposed approach in solving such equations, numerical examples are performed. The analysis of the obtained results shows that the FPSM is simple, straightforward and appropriate tool for solving various forms of these equations.

Keywords: Fractional power series method,fractional Volterra integro-differential equation, Caputo fractional derivative.

1. INTRODUCTION

In recent times, fractional Integro-differential equations (FIDEs) have played a vital role in the mathematical formulation of various problems that arise in fields of engineering and sciences, such as fluid dynamics, the theory of elasticity, electrodynamics, oscillating magnetic field, and so on [1-4]. The derivatives of fractional order provide an excellent tool in order to describe the memory and hereditary properties of different problems. Solving the FIDEs exactly is occasionally too complicated task, and hence finding good approximate and numerical solutions for this kind of equations using numerical methods will be very valuable.
Our concern in this work is to provide the analytic approximate solutions using fractional power series method (FPSM) for a class of fractional Volterra integro-differential equations(FVIDEs) of order 2β of the form:

$$\mathcal{D}_{t_0^+}^{2\beta} h(t) + h(t) = \int_{t_0}^{t} \omega(t,\xi)h(\xi)d\xi + f(t),\ \beta \in (0,1], \tag{1}$$

subject to initial condition

$$h(t_0) = h_0 \text{ and } \mathcal{D}_{t_0^+}^{\beta} h(t_0) = h_1. \tag{2}$$

where φ is a continuous function of$t$, $\omega(t,s)$ is called a crisp kernel function, $h_0, h_1 \in \mathbb{R}$ and the operator $\mathcal{D}_{t_0^+}^{(\cdot)}$ indicated to the Caputo derivative of fractional order in crisp sense. Here $h(t)$ is unknown function which needs to be determined.

Investigations of Volterra and Fredholm FIDEs by using different numerical methods are done by many experts. Among of these methods: variational iteration method [3];Adomian decomposition method [4];Spectral-collocation method [5];Homotopy perturbation method [6];Generalized Taylor matrix method [7].Further research papers regarding numerical techniques for integro-differential differential equations, we refer to [8-17].

This paper introduces a powerful analytical method, called fractional power series method (FPSM) for solving linear farctional Volterra integro-differential equations. This method

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combining of generalized Taylor formula and the concept of the residual error function under the Caputo meaning. The FPSM help us to obtain the approximate solutions in the form of convergent FPS without linearization, perturbation, or discretization [18-24]. It was applied successfully to solve different types of ordinary, fractional and fuzzy differential equations. The structured of this paper is as follow: In Section 2, some basic mathematical concepts are described. The analysis of the proposed method is given in Section 3. Simulations and test applications are performed to show the performance of the FPSM in Section 4. In Section 5, the conclusion is presented.

2. BASIC MATHEMATICAL CONCEPTS

The purpose of this section is to present some basic definitions and facts related to fractional calculus and fractional power series, which are used in this study [25-37].

Definition 2.1. The Caputo fractional derivative of a function h of order $\beta > 0$ is defined by:

$$\mathcal{D}_{t_0^+}^{\beta} h(t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_{t_0}^{t} \frac{h^{(m)}(\eta)}{(t-\eta)^{\beta-m+1}}\, d\eta, & m-1 < \beta < m, m \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} h(t), & \beta = m. \end{cases}$$

Definition 2.2. A fractional power series (FPS) representation at $t = t_0$ is given by

$$\sum_{j=0}^{\infty} h_j (t-t_0)^{j\beta} = h_0 + h_1 (t-t_0)^{\beta} + h_2 (t-t_0)^{2\beta} + \cdots,$$

where $0 \leq k-1 < \beta \leq k$ and $t \geq t_0$, and h_j's are the coefficients of the series.

Theorem 2.3. *Suppose that h has the following FPS representation at* t_0

$$h(t) = \sum_{j=0}^{\infty} h_j (t-t_0)^{j\beta}, \tag{3}$$

where $0 \leq k-1 < \beta \leq k, t \in [t_0, t_0 + R)$. *If* $h(t) \in C[t_0, t_0 + R$, *and* $\mathcal{D}_t^{j\beta} h(t) \in C(t_0, t_0 + R)$, *for* $j = 0,1,2,\ldots$, *then coefficients* h_j *will be in the form* $h_j = \frac{\mathcal{D}_t^{j\beta} h(t_0)}{\Gamma(j\beta+1)}$, *where* $\mathcal{D}_t^{j\beta} = \mathcal{D}_t^{\beta} \cdot \mathcal{D}_t^{\beta} \cdots \mathcal{D}_t^{\beta}$ *(j-times).*

Lemma 2.4. Suppose that $h(t) \in C[t_0, t_0 + R), R > 0, D_{t_0}^{j\beta} h(t) \in C(t_0, t_0 + R)$, and $0 < \beta \leq 1$. Then for any $j \in \mathbb{N}$, we have

$$\left(J_{t_0}^{j\beta} D_{t_0}^{j\beta} h\right)(t) - \left(J_{t_0}^{(j+1)\beta} D_{t_0}^{(j+1)\beta} h\right)(t) = \frac{D_{t_0}^{j\beta} h(t_0)}{\Gamma(j\beta+1)} (t-t_0)^{j\beta},$$

where $J_{t_0}^{j\beta}$ is the Riemann-Liouville fractional operator of order β.

Theorem 2.5. *Let* $h(t)$ *has the FPS in (3) with radius of convergence* $R > 0$, *and suppose that* $h(t) \in C[t_0, t_0 + R), D_{t_0}^{j\beta} h(t) \in C(t_0, t_0 + R)$ *for* $j = 0,1,2,\ldots,N+1$. *Then,*

$$h(t) = h_N(t) + R_N(\zeta), \tag{4}$$

where $h_N(t) = \sum_{j=0}^{N} \frac{D_{t_0}^{j\beta} h(t_0)}{\Gamma(j\beta+1)} (t-t_0)^{j\beta}$ *and* $R_N(\zeta) = \frac{D_{t_0}^{(N+1)\beta} h(\zeta)}{\Gamma((N+1)\beta+1)} (t-t_0)^{(N+1)\beta}$ *for some* $\zeta \in (t_0, t)$.

3. ANALYSIS OF PROPOSED ALGORITHM

In this section, we give the approximate solution of FVIDE (1) and (2) by means of the propsed method. The fractional power series (FPS) solution of (1) about $t_0 = 0$ has the following general form:

$$h(t) = \sum_{k=0}^{\infty} h_k \frac{t^{k\beta}}{\Gamma(k\beta + 1)}. \tag{5}$$

Subsequent, consider the nth-FPS solution by the following truncation series:

$$h_n(t) = \sum_{k=0}^{n} h_k \frac{t^{k\beta}}{\Gamma(k\beta + 1)}. \tag{6}$$

From initial condition (2), we have $h_1(t) = h_0 + h_1 \frac{t^{\beta}}{\Gamma(\beta+1)}$, which represents the first-FPS approximate solution of FVIDE (1) and (2). So, the expansion (5) will be written as

$$h_n(t) = h_0 + h_1 \frac{t^{\beta}}{\Gamma(\beta + 1)} + \sum_{k=2}^{n} h_k \frac{t^{k\beta}}{\Gamma(k\beta + 1)}. \tag{7}$$

Now, define the following residual function:

$$Res(t) = \mathcal{D}_{0^+}^{2\beta} h(t) + h(t) - \int_0^t \omega(t,\xi) h(\xi) d\xi - f(t). \tag{8}$$

Consequently, the nth-residual function is given by

$$Res_n(t) = \mathcal{D}_{0^+}^{2\beta} h_n(t) + h_n(t) - \int_0^t \omega(t,\xi) h_n(\xi) d\xi - f(t). \tag{9}$$

In order to find $h_2, h_3, h_4, \ldots$, we consider the nth-FPS solution for $n = 2,3,4, \ldots$ in (7),substitute it into (9), compute $\mathcal{D}_{0^+}^{(n-2)\beta}$ of the obtained equation and then lastly find the solution of $\mathcal{D}_{0^+}^{(n-2)\beta} Res_n(t)\Big|_{t=0} = 0, n = 2,3,4, \ldots$.

4. SIMULATION AND TEST APPLICATIONS

This section aims to test the validity and reliability of FPSM by applying it on two fractional integro-differential equations of Volterra type.

Application4.1 Consider the following fractional integro-differential equation of Volterra type:

$$\mathcal{D}_{0^+}^{2\beta} h(t) + h(t) = t + \cos(t) - \int_0^t (t - \xi) h(\xi) d\xi, \beta \in (0,1], t \geq 0, \tag{10}$$

with the initial conditions

$$h(0) = 0 \text{and } \mathcal{D}_{0^+}^{\beta} h(0) = 1. \tag{11}$$

The exact solution at $\beta = 1$ is $h(t) = \sin(t)$.

Following the procedure of RPS-algorithm, the FPS approximated solution of IVPs (10) and (11) has the form

$$h_n(t) = t + \sum_{k=2}^{n} h_k \frac{t^{k\beta}}{\Gamma(k\beta + 1)}. \tag{12}$$

Consequently, the nth-residual function is

$$Res_n(t) = \mathcal{D}_{0^+}^{2\beta}\left(t + \sum_{k=2}^{n} h_k \frac{t^{k\beta}}{\Gamma(k\beta+1)}\right) + \left(t + \sum_{k=2}^{n} h_k \frac{t^{k\beta}}{\Gamma(k\beta+1)}\right) + \int_0^t (t-\xi)\left(\xi + \sum_{k=2}^{n} h_k \frac{\xi^{k\beta}}{\Gamma(k\beta+1)}\right) d\xi - (t + \cos(t)). \quad (13)$$

Table 1: Numerical results for Example 4.1 at $\beta = 1$.

t	$h(t)$	$h_{10}(t)$	$\lvert h(t) - h_{10}(t)\rvert$
0.2	0.1986693307950612	0.1986693307936508	1.4104×10^{-12}
0.4	0.3894183423086505	0.3894183415873016	7.2135×10^{-10}
0.6	0.5646424733950355	0.5646424457142858	2.76807×10^{-8}
0.8	0.7173560908995228	0.7173557231746032	3.67725×10^{-5}

Table 2: Numerical results for Example 4.1 at different values of β.

t	10th RPS solutions			
	$\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
0.2	0.1986693308	0.2147230340	0.2499422596	0.2887197469
0.4	0.3894183416	0.4041866964	0.4319930895	0.4549562225
0.6	0.5646424457	0.5698565436	0.5735772157	0.5659522268
0.8	0.7173557232	0.7070947397	0.6766998551	0.6329008430

Numerical results for the 10th-approximated are given in Table 1 with step size 0.2 at $\beta = 1$. In which the 10th-approximated for different values of β, whereas $\beta = 0.95, \beta = 0.85$, and $\beta = 0.75$ are presented in Table 2. From these tables, it can be noted that the RPS method provides us with an accurate approximate solution, which is in good agreement with the exact solutions for all values of t in $[0,1]$.

CONCLUSION

In the present article, the analytic-numeric solution of linear fractional integro-differential equations of Volterrta type is constructed and analyzed by utilizing an efficient and accurate algorithm, named fractional power series algorithm. The FPS algorithm provides good analytic-numeric approximate solutions close to exact solutions. Two illustrative applications are tested to illustrate the accuracy and simplicity of the aforesaid method. Obtained results emphasized that the proposed method is a powerful and suitable technique to obtain the analytic-numeric approximate solutions for various types of fractional differential equations.

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SOLVING NONLINEAR FUZZY FRACTIONAL IVPS USING FRACTIONAL RESIDUAL POWER SERIES ALGORITHM

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ABSTRACT

Fuzzy initial value problems of fractional order play a vital role in modeling several realism matters arising in the natural sciences and engineering fields. In this paper, the fuzzy approximated solution of linear fuzzy fractional IVPs under the assumption of strongly generalized differentiability have been provided using fractional residual power series (FRPS) method. The solution methodology of the proposed algorithm depends on producing the solutions in r-cut representations with rapidly convergence fractional power series (FPS). Numerical problem is performed to demonstrate the accuracy, performance, and reliability of the present method. The effects of the fractional order α and the parameter r have been shown graphically and quantitatively. The results obtained indicate to an agreement well between the fuzzy exact and fuzzy approximated solutions, as well as satisfy the symmetry convex triangular fuzzy number. Therefore, the FRPS method is an accurate, effective, simple and suitable tool to apply in finding the solutions of such problems.

Keywords: Fuzzy number, fractional residual power series method, fuzzy fractional initial value problems, strongly generalized differentiability

1. INTRODUCTION

Fuzzy differential equations (FDEs), being a significant area of study the behavior of dynamical systems, has captured the interest of several scientists during past decades. It has wide applications in various and engineering and physical processes [1-8]. The starting point of the fuzzy derivative was introduced by Chang et al. [9], and then Dubois et al. [10] have used the extension principle in their approach. Later on, Puri and Ralescu [11] developed the derivative for fuzzy-valued mappings by generalized and extended the concept of Hukuhara differentiability for set-valued functions to the class of fuzzy functions. Subsequently, Kaleva [12] and Seikkala [13] started using the Hukuhara derivative to develop the theory of fuzzy differential equations.

This article purposes to employed an numerical-analytical recent approach, called fractional residual power seies (FRPS) algorithm for solving the following fuzzy fractional IVPs

$$\begin{cases} D_{0^+}^{\alpha}u(x) = F\big(x, u(x)\big), 0 < \alpha \leq 1, x \in [0,1] \\ \qquad\qquad u(0) = u_0 \end{cases}, \tag{1}$$

where $D_{0^+}^{\alpha}$ is the fuzzy Caputo fractional derivative of order α, $F: [0,1] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a continuous nonlinear fuzzy-valued function, $u_0 \in \mathbb{R}_{\mathcal{F}}$ and $u(x)$ is unknown analytical function to be determined.

The fractional residual power series (FRPS) method is a numeric-analytic method for solving different types of ordinary, partial, and fuzzy differential equations of arbitray order. The starting point of this method had been presented by Abu Arqub in [14]. The methodology of the FRPS approachgives a Maclaurin expansion of the solution based on the Caputo sense [15-24]. Throughout this article $\mathbb{R}_{\mathcal{F}}$ denote the set of fuzzy numbers on $\mathbb{R}$.

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2. PRELIMINARIES AND NOTATIONS

Definition 2.1. Suppose that φ is a fuzzy subset of $\mathbb{R}$. Then, φ is called a fuzzy number such that φ is upper semicontinuous membership function of bounded support, normal, and convex.

Definition 2.3 The complete metric structure on $\mathbb{R}_{\mathcal{F}}$ is given by the Hausdorff distance mapping $D_{\mathcal{H}}: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^{+} \cup \{0\}$ such that $D_{\mathcal{H}}(\varphi, \omega) = sup_{0 \leq r \leq 1} max\{|\varphi_{1r} - \omega_{1r}|, |\varphi_{2r} - \omega_{2r}|\}$, for arbitrary fuzzy numbers $\varphi = (\varphi_1, \varphi_2)$ and $\omega = (\omega_1, \omega_2)$.

Definition 2.4. For fixed $x_0 \in [a,b]$ and $u: [a,b] \to \mathbb{R}_{\mathcal{F}}$, the function u is called a strongly generalized differentiable at x_0, if there is an element $u'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that either:

1) The H-differences $u(x_0 + \varepsilon) \ominus u(x_0)$, $u(x_0) \ominus u(x_0 - \varepsilon)$ exist, for each $\varepsilon > 0$ sufficiently tends to 0 and $\lim_{\varepsilon \to 0^+} \frac{u(x_0 + \varepsilon) \ominus u(x_0)}{\varepsilon} = u'(x_0) = \lim_{\varepsilon \to 0^+} \frac{u(x_0) \ominus u(x_0 - \varepsilon)}{\varepsilon}$,
2) The H-differences $u(x_0) \ominus u(x_0 + \varepsilon), u(x_0 - \varepsilon) \ominus u(x_0)$ exist, for each $\varepsilon > 0$ sufficiently tends to 0 and $\lim_{\varepsilon \to 0^+} \frac{u(x_0) \ominus u(x_0 + \varepsilon)}{-\varepsilon} = u'(x_0) = \lim_{\xi \to 0^+} \frac{\varphi(x_0 - \varepsilon) \ominus \varphi(x_0)}{-\varepsilon}$,

where the limit here is taken in the complete metric space $(\mathbb{R}_{\mathcal{F}}, D_{\mathcal{H}})$.

Definition 2.5. Let $u: [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $u \in C^{\mathcal{F}}[a,b] \cap L^{\mathcal{F}}[a,b]$. One can say u is Caputo fuzzy H-differentiable at x when $D_{a^+}^{\alpha} u(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{u'(\tau)}{(x-\tau)^{\beta}} d\tau$ exists, where $0 < \alpha \leq 1$. As well, we say that u is Caputo$[(1) - \alpha]$differentiable if u is (1)-differentiable and u is Caputo $[(2) - \alpha]$differentiable if u is (2)-differentiable.

Definition 2.6. A fractional power series (FPS) representation at $x = a$ has the following form
$$\sum_{k=0}^{\infty} u_k (x-a)^{k\alpha} = u_0 + u_1(x-a)^{\alpha} + c_2(x-a)^{2\alpha} + \cdots,$$
where $0 \leq n - 1 < \alpha \leq n$ and $x \geq a$, and u_k's are the coefficients of the series.

Theorem 2.7.*Suppose that $f(x)$ has the following FPS representation at $x = a$*
$$f(x) = \sum_{k=0}^{\infty} u_k (x-a)^{k\alpha}, \quad m - 1 < \alpha \leq m, a < x < a + R.$$
where $f(x) \in C[a, a+R)$ and $D_{a^+}^{k\alpha} f(x) \in C(a, a+R)$ for $k = 0,1,2, \ldots$, then the coefficients u_k will be in the form $u_k = \frac{D_{a^+}^{k\alpha} f(a)}{\Gamma(k\alpha+1)}$such that $D_{a^+}^{k\alpha} = D_{a^+}^{\alpha} \cdot D_{a^+}^{\alpha} \cdot \ldots \cdot D_{a^+}^{\alpha}$ (k-times).

3. FUZZY FRACTIONAL INITIAL VALUE PROBLEMS

The (n)-solution of FFIVPs (1) is a function $u: [0,1] \to \mathbb{R}_{\mathcal{F}}$ that has Caputo $[(n) - \alpha]$-differentiable and satisfies the FFIVPs (1). To compute it, the next algorithm show us the strategy to solve the FFIVPs (1) in parametric form in term of its r-levels representation. Indeed, there are two cases depends on differentiability type [25-32].

Algorithm 3.1: To determine the (n)-solutions of FFIVPs (1), there are two cases:

Case (I): If $u(x)$ is Caputo [(1)- α]-differentiable,the FFIVPs (1) will be converted to the followingcrispsystem; Then, do the following steps:

$$\begin{cases} D_{0^+}^{\alpha} u_{1r}(x) = F_{1r}(x, u_{1r}(x), u_{2r}(x)) \\ D_{0^+}^{\alpha} u_{2r}(x) = F_{2r}(x, u_{1r}(x), u_{2r}(x)), \\ u_{1r}(0) = u_{01r}, \text{and } u_{2r}(0) = u_{02r} \end{cases} \tag{3}$$

Step1: Solve the system for $u_{1r}(x)$ and $u_{2r}(x)$
Step2: Ensure $[u_{1r}(x), u_{2r}(x)]$, and $[D_{0^+}^{\alpha} u_{1r}(x), D_{0^+}^{\alpha} u_{2r}(x)]$are valid level sets for $r \in [0,1]$.
Step3: Construct the (1)-solution, $u(x)$ whose r-level representation $[u_{1r}(x), u_{2r}(x)]$.

Case (II): If $u(x)$ is Caputo [(2)- α]-differentiable,the FFIVPs (1) will be converted tothe following crisp system; Then, do the following steps:

$$\begin{cases} D_{0^+}^{\alpha}u_{1r}(x) = F_{2r}(x, u_{1r}(x), u_{2r}(x)) \\ D_{0^+}^{\alpha}u_{2r}(x) = F_{1r}(x, u_{1r}(x), u_{2r}(x)), \\ u_{1r}(0) = u_{01r}, \text{and } u_{2r}(0) = u_{02r} \end{cases} \tag{4}$$

Step1: Solve the system for $u_{1r}(x)$ and $u_{2r}(x)$
Step2: Ensure $[u_{1r}(x), u_{2r}(x)]$, and $\left[D_{0^+}^{\alpha}u_{2r}(x),\ D_{0^+}^{\alpha}u_{1r}(x)\right]$are valid level sets for $r \in [0,1]$.
Step3: Construct the (2)-solution, $u(x)$ whose r-level representation $[u_{1r}(x), u_{2r}(x)]$.

4. DESCRIPTION OF FRPS METHOD

In this section, we show the basic idea of the FRPS method in finding the(1)-solution for the system of OFDEs (3). In the same manner,we can apply the propsed method to construct (2)-solution of (4). To reach our purpose, we suppose that the solutions of (3)about$a = 0$are given by

$$\begin{aligned} u_{1r}(x) &= \textstyle\sum_{j=0}^{\infty} \gamma_j x^{j\alpha}, \\ u_{2r}(x) &= \textstyle\sum_{j=0}^{\infty} \mu_j x^{j\alpha}. \end{aligned} \tag{5}$$

Using the conditions (3), the approximate values of (5) can be found usingm^{th}-truncated series:

$$\begin{aligned} u_{m,1r}(x) &= \gamma_0 + \textstyle\sum_{j=1}^{m} \gamma_j x^{j\alpha}, \\ u_{m,2r}(x) &= \mu_0 + \textstyle\sum_{j=1}^{m} \mu_j x^{j\alpha}. \end{aligned} \tag{6}$$

In order to determine the unknown coefficients $\gamma_j$and μ_j for $j = 1,2, \dots, m$, we define the following m^{th}-residual functions:

$$\begin{aligned} Res_{m,1r}(x) &= D_{0^+}^{\alpha}u_{m,1r}(x) - F_{1r}\big(x, u_{m,1r}(x), u_{m,2r}(x)\big), \\ Res_{m,2r}(x) &= D_{0^+}^{\alpha}u_{m,2r}(x) - F_{2r}\big(x, u_{m,1r}(x), u_{m,2r}(x)\big). \end{aligned} \tag{7}$$

From (6), we notice that $\lim_{m\to\infty} Res_{m,nr}(x) = Res_{nr}(x) = 0$, for each $x \geq 0$ and $n \in \{1,2\}$, which leads $D_{0^+}^{(k-1)\alpha}Res_{m,nr}(x) = 0$. Furthermore, $D_{0^+}^{(m-1)\alpha}Res_{nr}(0) = D_{0^+}^{(m-1)\alpha}Res_{m,nr}(0) = 0$, for each $m = 1,2, \dots, j$.

5. Numerical Experiment

In this section, we consider a linear fuzzy fractional initial value problem to illustrate the efficiency and applicability of the FRPS algorithm. Here, all the symbolic and numerical computations performed by using Mathematica 10.

Example 5.1 Consider the following linear fuzzy fractional initial value problem:

$$\begin{cases} D_{0^+}^{\alpha}u(x) = [r+1, 3-r] + u(x), x \in [0,1], 0 < \alpha \leq 1, \\ \qquad u(0) = 0. \end{cases} \tag{8}$$

Based on the type of differentiability, then the FFIVPs (8) can be converted to the following systems:
Case1: Under Caputo [(1)- α]-differentiability, the system of OFDEs is given by

$$\begin{cases} D_{0^+}^{\alpha}u_{1r}(x) - u_{1r}(x) - (r+1) = 0 \\ D_{0^+}^{\alpha}u_{2r}(x) - u_{2r}(x) - (3-r) = 0, \\ \qquad u_{1r}(0) = u_{2r}(0) = 0 \end{cases} \tag{9}$$

In view of the last discussion for the FRPS algorithm, starting with $u_{0,1r}(0)=0$, and $u_{0,2r}(0)=0$ and depend on the result $D_{0^+}^{(m-1)\alpha}Res_{k,1r}(0)=D_{0^+}^{(m-1)\alpha}Res_{k,2r}(0)=0, m=1,2,\dots,6$, the 6th-FRPS approximated solutions for (9) are given by

$$u_{6,1r}(x)=(r+1)\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}\right),$$
$$u_{6,2r}(x)=(3-r)\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}\right).$$

Hence, the approximated solutions for OFDEs (9) can be written as

$$u_{1r}(x)=(r+1)\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\cdots\right),$$
$$u_{2r}(x)=(3-r)\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\cdots\right).$$

which are coincide well with the Taylor series expansion of the exact solution $[u(x)]^{\mathrm{r}}=[r+1,3-r](e^{x}-1)$ when $\alpha=1$.

Case2: Under Caputo [(2)- α]-differentiability, the system of OFDEs is given by

$$\begin{cases} D_{0^+}^{\alpha}u_{1r}(x)-u_{2r}(x)-(3-r)=0 \\ D_{0^+}^{\alpha}u_{2r}(x)-u_{1r}(x)-(r+1)=0, \\ \quad u_{1r}(0)=u_{2r}(0)=0 \end{cases} \tag{10}$$

By FRPS-algorithm, the 6th-FRPS approximated solutions for OFDEs (10) are given by

$$u_{6,1r}(x)=\left(\frac{(3-r)x^{\alpha}}{\Gamma(\alpha+1)}+\frac{(r+1)x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(3-r)x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{(r+1)x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{(3-r)x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{(r+1)x^{6\alpha}}{\Gamma(6\alpha+1)}\right),$$
$$u_{6,2r}(x)=\left(\frac{(r+1)x^{\alpha}}{\Gamma(\alpha+1)}+\frac{(3-r)x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(r+1)x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{(3-r)x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{(r+1)x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{(3-r)x^{6\alpha}}{\Gamma(6\alpha+1)}\right).$$

Thus, the approximated solutions for OFDEs (10) can be expressed as

$$u_{1r}(x)=\left(\frac{(3-r)x^{\alpha}}{\Gamma(\alpha+1)}+\frac{(r+1)x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(3-r)x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{(r+1)x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{(3-r)x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{(r+1)x^{6\alpha}}{\Gamma(6\alpha+1)}+\cdots\right),$$
$$u_{2r}(x)=\left(\frac{(r+1)x^{\alpha}}{\Gamma(\alpha+1)}+\frac{(3-r)x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(r+1)x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{(3-r)x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{(r+1)x^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{(3-r)x^{6\alpha}}{\Gamma(6\alpha+1)}+\cdots\right).$$

which are coincide well with the Taylor series expansion of the exact solution $[u(x)]^{r}=2e^{x}+[1-r,r-1](1-e^{-x})$ when $\alpha=1$.

To show the accurecy and efficiency of the method. The absolute errors of $u_{1r}(x)$ and $u_{2r}(x)$ have been obtainedin Table 1 at $\alpha=1$ for different values of r, FFIVPs (9), case 1.

Table 1: Absolute errors of $u_{1r}(x)$ and $u_{2r}(x)$ at $\alpha=1$ and $n=8$, for Example 5.1, case 1.

	$u_{1r}(x)$		
x_i	$r=0$	$r=0.5$	$r=1$
0.16	0.0	0.0	0.0
0.32	1.000×10^{-10}	0.0	3.00×10^{-10}
0.48	4.000×10^{-9}	6.000×10^{-9}	8.00×10^{-9}
0.64	5.300×10^{-8}	7.900×10^{-8}	1.06×10^{-7}
0.80	4.020×10^{-7}	6.030×10^{-7}	8.04×10^{-7}
0.96	2.109×10^{-6}	3.164×10^{-6}	4.21×10^{-6}
	$u_{2r}(x)$		
x_i	$r=0$	$r=0.5$	$r=1$
0.16	0.0	0.0	0.0
0.32	4.00×10^{-10}	0.0	3.00×10^{-10}
0.48	1.20×10^{-8}	1.00×10^{-8}	8.00×10^{-9}
0.64	1.59×10^{-7}	1.33×10^{-7}	1.06×10^{-7}
0.80	1.20×10^{-6}	1.00×10^{-6}	8.04×10^{-7}

0.96	6.32×10^{-6}	5.27×10^{-6}	4.21×10^{-6}

As well, we have been given in Table 2, the numerical results of th 8th-FRPS approximated solutions, for case 2 at different values of α and r with some selected grid points on $[0,1]$.

Table 2: Approximated solutions of $u_{1r}(x)$ and $u_{2r}(x)$, at $n = 8$, for Example 5.1, case 2.

$u_{1r}(x)$					
r_i	x_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
0.5	0.2	0.53344013	0.66818510	0.83949977	1.06177122
	0.4	1.14848937	1.36694023	1.63804155	1.98417478
	0.6	1.41864337	2.17213346	2.54636969	3.02406979
	0.8	2.17574570	3.12343484	3.61608252	4.24615929
1	0.2	0.44280551	0.56106631	0.71481241	0.91898984
	0.4	0.98364939	1.18748975	1.44521580	1.77954105
	0.6	1.64423754	1.93769061	2.30515534	2.77803422
	0.8	2.45108105	2.84587694	3.33826533	3.96963386
$u_{2r}(x)$					
r_i	x_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
0.5	0.2	0.35217089	0.45394752	0.59012504	0.77620845
	0.4	0.81880941	1.00803927	1.25239005	1.57490781
	0.6	1.41864337	1.70324777	2.06394099	2.53200698
	0.8	2.17574570	2.56831904	3.06044813	3.69317084
1	0.2	0.44280551	0.56106631	0.71481241	0.91898984
	0.4	0.98364939	1.18748975	1.44521580	1.77954105
	0.6	1.64423754	1.93769061	2.30515534	2.77803422
	0.8	2.45108105	2.84587694	3.33826533	3.96963386

CONCLUSION

In this paper, the fractional residual power series algorithm has been applied to investigate the solution of linear FFIVPs under the assumption strongly generalized differentiability. The present algorithm gives accurate and efficient analytical solutions without require being linearized, discretized or perturbation. From obtained results, the fuzzy approximated solutions are coinciding well with each other, and with the fuzzy exact solution as well indicate that the proposed approach is a direct, simple, and very convenient algorithm to solve such problems and suitable to deal with a wide variety of other fuzzy differential equations of fractional order.

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OPTIMUM STRATUM BOUNDARIES USING ARTIFICIAL BEE COLONY AND PARTICLE SWARM OPTIMIZATION

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ABSTRACT

Optimum stratification is the method of choosing the best boundaries to make strata homogeneous .It is used to attain more precision and accuracy than other methods of sampling . The main idea behind this method is that a heterogeneous population is partitioned into subpopulations, each of which is internally homogeneous. The main obstacle associated with stratified sampling is how to gain the optimum boundaries with minimum variance .It is well known that several numerical and computational methods have been changed for this goal , some of them are designed to highly skewed populations and others to any kind of populations This paper considers an Artificial Bee Colony (ABC) algorithm to arrive at the optimum of stratum boundaries depending on Neyman Allocation The ABC algorithm is used on two groups of populations and a comparative study with Particle Swarm Optimization (PSO) is given . The paper concludes that numerical results show that the proposed algorithm is able to find the optimum stratum boundaries for a set of standard populations and various standard test functions compared with (PSO) algorithms.

***Keywords*:**Stratified random sampling; Neyman Allocation; Artificial Bee Colony ; Particle Swarm Optimization Optimum Stratum Boundaries

1. INTRODUCTION

Stratified random sampling or proportional random sampling is a commonly used sampling method especially for heterogeneous populations. Stratified sampling is preferably chosen for its capability of improving statistical accuracy resulting in a smaller variance of the estimator, in comparison with simple random sampling. In order to decrease the variance of the estimator in stratified sampling [2].

Several numerical and computational methods have been invented to achieve the optimal limits in class sampling. Some apply to highly deviant populations while others apply to any type of population. A very early and simple method is the cumulative square root of the cumff method of Dalenius & Hodges in 1959 [6]. We also propose the Lavallée and Hidiroglou [13] algorithm for highly skewed groups, while Kozak (2004) [12] and the Kennedy & Eberhart method in 2001 (pso) [9] were preferred to non-perverted populations.

This study proposes the ABC algorithm for defining stratum boundaries. In order to find out the efficiency of ABC algorithm,it is compared with Particle Swarm Optimization (PSO)

2. STRATIFIED RANDOM SAMPLING

The equal allocation method is considered the simplest one where each stratum sample has the same size. With the Neyman allocation method, the sample size in each stratum follows Neyman allocation.[14]

we have each character expresses the value as follows: Y:stratification variable; N:population size; n: sample size; L: number of strata; Nh: number of elements in stratum h(h = 1, . . . , L);nh: sample size in stratum h; : mean of elements in stratum h; estimated mean in stratified sampling :variance of the estimated mean in stratified sampling .[1]

In stratified sampling [5], a population with N units is separated into L groups with N1,N2, ...,Ni, ...,NL units respectively. These groups are called strata. like that

1 + N2 + ... + Nh + ... + NL= N ……….(1)

We also have a variance in the mean of the stratified sample is:

$$V(\bar{y}_{st}) = \sum_{h=1}^{L} W^2{}_h \frac{\sigma^2{}_h}{n_h} \qquad \ldots\ldots\ldots\ldots(2)$$

The equation for Neyman allocation is written as follows .

$$V_{Ney\,(\bar{y}_{st})} = \frac{1}{n}\left(\sum_{h=1}^{L} W_h \sigma_h\right)^2 \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

3. WHAT IS THE ARTIFICIAL BEE COLONY ALGORITHM

The artificial bee colony algorithm was proposed by Karaboga in "2005" to develop the digital function .It simulates the colony of bees depending on the intelligence of a swarm. The following are some of the main steps of the artificial bee colony algorithm[7]

The colony has three kinds of bees: employed bees, onlooker bees and scout bees. Employed bees cover half the colony , and the other half is onlooker bees. The employed bees search for the food source and send the information of the food source to the onlooker bees. The onlooker bees choose a food source to exploit the information shared by the employed bees. The scout bee, which is one of the employed bees whose food source are abandoned, finds a new food source randomly. We can employ and adopt food source as a solution for development. Denote the food source number as *SN*, the position of the i^{th} food source as x_i(i =1,…, *SN*), which is a *D* dimensional vector [8,11].

In ABC algorithm, the ith fitness value i fit for a minimization problem is defined as[10]:

$$\text{fitness}_i = \begin{cases} 1/(1+f_i) & \text{if } f_i \geq 0 \\ 1+\text{abs}(f_i) & \text{if } f_i < 0 \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where(f_i) is the cost value of the i^{th} solution. The probability that food source being selected by an onlooker bee is given by:

$$\left.\begin{array}{l} p_i = \text{fitness}_i / \sum_{i=1}^{SN} \text{fitness}_i \\ p_i = ((0.9)\,\text{fitness}_i / \max(\text{fitness}_i)) + 0.1 \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

A candidate solution from the old one can be generated as:

$$v_{ij} = x_{ij} + \phi_{ij}(x_{ij} - x_{kj}) \qquad \ldots\ldots\ldots\ldots(6)$$

Where $k \in \{1,2,\ldots,SN\}$ and $j \in \{1,2,\ldots,D\}$ are randomly selected indices,ϕ_{ij} [-1, 1] is a uniformly distributed random number. The candidate solution is compared with the old one, and the better one should be remained [8].*If the abandoned food source is x_i, the scout bee exploits a new food source according to:

$$x_{ij} = x_{min,j} + \text{rand}(0,1)(x_{max,j} - x_{min,j}) \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

Where $x_{max,j}$ and $x_{max,j}$ are the upper and lower bounds of the j^{th} dimension of the problem's search space [11].

4. SEARCH MECHANISM

The exploration and exploitation abilities are essential for the population based algorithms، So it is very important to balance these two abilities to gain good optimization performance .

The modified search equation in onlooker bee stage is described as follows[9]:

$v_{ij} = x_{ij} + \phi_{ij}(x_{ij} - x_{kj}) + \vartheta_{ij}(y_i - x_{ij})$ .... (8)

Where k {1,2,..., SN} is a random selected index which differs from $i \in \{1,2,\dots,SN\}$, $j \in \{1,2,\dots,D\}$ is a random selected index, y_j is the j^{th} element of the global best solution,[

$\phi_{ij} = (-1,1)$, $\vartheta_{ij} \in (0,1.5)$, are both uniformly distributed random numbers.

by "DE/current-to-rand/1" [4] mutation strategy and based on the property of ABC algorithm, a new search equation in employed bee stage is proposed as follows:

$v_{ij} = x_{i,j} + \phi_{ij}(x_{ij} - x_{kj}) + \vartheta_{ij}(x_{r1,j} - x_{r2j})$ ....(9)

Where $\phi_{ij} = (-1,1)$, $\vartheta_{ij} \in (0,0.5)$

$\&i \in \{1,2,\dots,SN\}$, $j \in \{1,2,\dots,D\}$, $r1 \in \{1,2,\dots,SN\}$ and $r1 \neq r2 \neq i$

More easily and clearly, the new research equation and research mechanism is proposed to balance exploration capacity and utilization capacity in the ABC algorithm.

5. NUMERICAL EXPERIMENTS

The ABC experiments for the stratification sampling has been on populations data and functions, to find optimal strata boundaries based on variance of Neyman allocation. All experiments are implemented using Matlab (R2018b).

5.1 tasting artificial bee colony algorithm to find stratified boundaries.

We test the ABC algorithm and compare it with previous results for the POS algorithm[3]. Some groups are used for class, central, standard deviation and size. Each population is divided into 3, 4, 5 and 6 strata . The function uses probability density and is divided into 2, 3, 4, 5 strata.

These populations and function are:

Pop1: The population in thousands of US cities in 1940 (US cities).

Pop2: Central of banks in Iraq(2010)(CBI)

F(x) =2(1-x) ‘‘‘‘‘‘ Range $0 \le x \le 1$

Table 1: Comparison results for pop1 and pop 2

ABC	PSO	H
Pop1 : US cities		
V_{ney}	V_{ney}	
0.891951	0.891952	3
0.472274	0.472761	4
0.264202	0.264204	5
0.194225	0.196972	6
Pop2: CBI		
7.8349e+06	7.7133e+08	3
3.7039e+06	3.7770e+08	4
2.5576e+06	2.5664e+08	5
1.9558e+06	1.9635e+08	6

Table 2 : The Comparison results for the probability density functions using four different strata

ABC		*PSO*		L
Strata Boundaries	V_{ney}	Strata Boundaries	V_{ney}	
0.3542	0.0150372 0	0.354	0.0152	***2***
0.2298 0.5026	0.0068784 2	0.229 0.502	0.0069	***3***
0.1703 0.3606 0.5869	0.0039171 5	0.170 0.362 0.587	0.0039 2	***4***
0.1358 0.2833 0.4480 0.6432	0.0025363 4	0.135 0.282 0.447 0.642	0.0026	***5***

CONCLUSIONS

The numerical results emphasize the efficiency and capabilities of ABC algorithm in finding the Optimal Strata Boundaries. Amazingly, its performance seems better than PSO method This confirms that ABC can be efficiently utilized in the stratification of heterogeneous populations.

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FITTING STRUCTURAL MEASUREMENT ERROR MODELS USING REPETITIVE WALD-TYPE PROCEDURE

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ABSTRACT

In this paper, fitting structural regression model when both variables are subject to error is considered using a new estimation procedure. The new estimation procedure is a repetitive procedure extension to the Wald estimation method. A Monte Carlo experiments are conducted to study the performance of the new estimators and the results are compared with the classical two-group and three- group estimators in terms of the mean squared error. Moreover, a real data analysis to study the relationship between the human development indexand the national gross domestic productis discussed.

Keywords: Error-in-Variables Model; Wald Estimators;Human Development Index.

1. INTRODUCTION

Structural Measurement Error Model (MEM)[14,17] is an extension of the simple linear regression by assuming dependent and independent variables are measured by error. The corresponding standard linear MEM [13] assumes that two mathematical variables ξ and η are related as

$$\eta = \alpha + \beta_1 \xi$$

where the variables ξ and η are unobservable and can only be observed with additive errors as

$$x = \xi + \delta \text{ and } y = \eta + \varepsilon$$

assuming that the ξ and the errors terms, δ and ε, are uncorrelated. For a random sample of size n, say (x_i, y_i), $i = 1,2,....,n$; the structural MEM [12] can be formulated as

$$\left.\begin{aligned} &\eta_i = \alpha + \beta\,\xi_i \quad , i = 1,2,.....,n \\ &\text{where} \\ &x_i = \xi_i + \delta_i \text{ , and } \; y_i = \eta_i + \varepsilon_i \; i = 1,2,....,n \end{aligned}\right\} \quad (1)$$

The main problem in Eq.(1)is to estimate the unknown parameters $\alpha and \beta$.Several authors have discussed a couple of estimation methods to fit the structural MEM.Moreover, there are two types of estimation methods: parametric and non-parametric.For the parametric estimation method, the method of choice will be the maximum likelihood estimation method proposed by Lindley[18]which solves the problem by adding prior assumptions. Madansky [20] wrote a detailed summary on the problem of fitting a straight line using MLE when both variables are subject to error. Thompson and Carter [23] introduced an overview of the normal theory structural measurement error models.Cao et al. [9]have proposed of using an empirical Bayes approach by considering the EM algorithms to calculate maximum likelihood estimates for the MEM with or without equation error. Cao et al.[8]have obtained iterative formulas of maximum likelihood estimations via EM algorithm for the Heteroscedastic MEM.For the non-parametrictype of estimation method, Wald type estimation methods of the socalled grouping methodswere proposed by Wald [24]and modified by Nair and Shrivastava[21]. Recently, the information theory was used byAl-Nasser[2, 3]. Other authors like Al-Nasser [5] and Carroll [11] have proposed a non-parametric estimatorof a regression function from data that are impure by a mixture of the two errors (classical and Berkson).Moreover, robust non-parametric estimation procedures were proposed by Al-Nasser [4,6]and Wiedermann et al.[25]. More details about different estimation methods on the MEM context can be found in [1, 7, 10, 15, 16, 19, 22].

In this paper, a new non parametric estimation procedure is proposed and discussed numerically.This paper is divided into six sections. The second section is designated to review the classical Wald-type estimation methods. The third section introduces the new idea of an estimation procedure. The Fourth section presents Monte Carlo experiment to study the performances of the proposed estimators in fitting the MEM. The fifth section includes a real data analysis to study the relationships between Human development index (HDI)and the national gross domestic product (GDP); and the paper ends withthe sixth which presents concluding remarks.

2. THE CLASSICAL WALD TYPE ESTIMATION METHODS

The idea of the Wald type estimation methods as given by Gillard [15] and Wald[24] suggests of splitting the observations into two groups namely; "G1 and G2" of the same size (say *m*). Such that G1 contains the first half of the ordered observations ($X_{(1)}$, $Y_{(1)}$), ………, ($X_{(m)}$, $Y_{(m)}$)) and G2 contains the second half ($X_{(m+1)}$, $Y_{(m+1)}$), ………, ($X_{(n)}$, $Y_{(n)}$)). Then finds the slope between the central tendency of these groups. To be more clear, the steps of Two-Group estimation method are:

- Order the data based on X's values from smallest to largest.
- Divide the sample into two equal groups.
- Note: If we have an odd sample size, then remove median.
- Select the associated Y's values of X's.
- Compute the average of each sub-group.
- The point estimators aregiven in Eq.(2) and Eq.(3):

$$\hat{\beta} = \frac{\bar{y}_{G2}-\bar{y}_{G1}}{\bar{x}_{G2}-\bar{x}_{G1}} \quad (2)$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \quad (3)$$

Where$\bar{y}_{G2}$ = sample mean for *y* values in G2; $\bar{y}_{G1}$ = sample mean for *y* values in G1.$\bar{x}_{G2}$= sample mean for *x* values in G2.; $\bar{x}_{G1}$= sample mean for *x* values in G1.An extension of the two group procedure was proposed by Bartlett [7] and Nair and Shrivastava [21], by suggesting of splitting the observations into three equally sub-groups, "G1, G2 and G3"; and discard the middle group from the analysis.

3. THE PROPOSED ESTIMATION METHOD

The proposed estimation method is an extension of the classical Wald type procedure. It is a repetitive procedure depend on sorting the observed pairs (x_i, y_i)'s, i = 1, 2,…,n; by the extent of x_i's, then split the observation into several groups (say, r) of the same size and then find all possible paired slopes. The procedure can be described as follows:

- Order the x's data from smallest to largest and take the associated y's valued
- Divide the data into *r*-subgroups each of size k; where $r \leq [\frac{n}{2}]$.
- Compute the central tendency measure for each subgroup,
- Define the jth slope as follows:

$$\hat{\beta}_j = \frac{\bar{y}_{nj}-\bar{y}_{mj}}{\bar{x}_{nj}-\bar{x}_{mj}} \qquad j = 1, 2, \ldots\ldots, \binom{r}{2}, \qquad n, m = 1, 2, \ldots\ldots r, \text{ and } m < n$$

- The final estimators estimator will be as given in Eq.(4)

$$\hat{\beta} = \frac{1}{\binom{r}{2}} \sum_j \hat{\beta}_j \quad \text{and} \quad \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \quad (4)$$

4. MONTE CARLO EXPERIMENT

To study the performance of the proposed methods, two random samples were studied inlier and outlier samples based on a 10000 random samples each of size n that generated from the standard normalMEMas given in Eq.(1); under the following assumptions:

(1) The parameter initial values are ($\alpha = 0$, $\beta = 1$, $\sigma_\epsilon^2=1$, $\sigma_\delta^2=1$and $\sigma_\xi^2=1$)
(2) Three different sample sizes are considered; n = 10, 50 and 100.
(3) For the Proposed procedure, the sample suggested to be divided into r = 3, 4 samples.
(4) For the outlier case, the data was contaminated; at each step a certain percentage (10%) of the observations were deleted and replaced with outliers' observations. The contaminated data point was generated according to the given relationship where:

    (i) In y only outliers ($\epsilon_i \sim N(0, \sigma_\epsilon^2)$),$\sigma_\epsilon^2 = 16$.
    (ii) In x only outliers ($\delta_i \sim N(0, \sigma_\delta^2)$), $\sigma_\delta^2 = 16$.
    (iii) In both x and y outliers (σ_ϵ^2,σ_δ^2) = (16, 16).

The performances of these estimators were measured by using the simulated bias and mean square error:

$$Bias = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\mu}_i - \mu); \quad MSE = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\mu}_i - \mu)^2$$

Where $\hat{\mu}_i$ is the estimates given by one of the proposed estimators for the i^{th} sample.The Monte Carlo experiment results are given in Table.1 for inliers cases; however, Table 2 Table 3 and Table 4 for outlier in x only, outlier in y only, outliers in both x and y, respectively. The simulated results indicated that, the classical Wald type estimation procedure is better than the proposed procedure when the sample size is small (n = 10). Then as increasing the sample size the proposed procedure robustify the classical Wald type procedure in terms of the Bias and the MSE for both parameters.

Table 1: Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$: Inlier case.

n	Parameter	Statistic	Estimation Methods			
			Two group	Three group	Repetitive r = 3	Repetitive r = 4
10	$\hat{\alpha}$	Bias	0.0051	-0.0055	0.4253	-0.0146
		MSE	0.01821	0.02888	0.05473	0.02865
	$\hat{\beta}$	Bias	-0.4969	-0.5005	-0.4783	-0.4852
		MSE	0.04167	0.04085	0.04278	0.06391
50	$\hat{\alpha}$	Bias	0.0026	0.0067	0.4847	0.0027
		MSE	0.00061	0.00098	0.00575	0.00065
	$\hat{\beta}$	Bias	-0.499	-0.4979	-0.4937	-0.5002
		MSE	0.0055	0.00538	0.00531	0.00545
100	$\hat{\alpha}$	Bias	0.0015	-0.0006	0.4951	0.0003
		MSE	0.00015	0.00024	0.00271	0.00015
	$\hat{\beta}$	Bias	-0.4996	-0.4997	-0.4978	-0.4997
		MSE	0.00262	0.002599	0.002577	0.002597

Table 2: Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$: outlier in x.

σ_δ^2	n	Estimate	Statistic	Estimation Methods			
				Two group	Three group	Repetitive r = 3	Repetitive r = 4
16	10	$\hat{\alpha}$	Bias	-0.0086	-0.002	0.6636	0.0014
			MSE	0.02864	0.05082	0.11561	0.14237

	n	Estimate	Statistic	Two group	Three group	Repetitive r = 3	Repetitive r = 4
		$\hat{\beta}$	Bias	-0.7539	-0.7753	-0.7099	-0.6876
			MSE	0.06365	0.0666	0.06218	0.07263
	50	$\hat{\alpha}$	Bias	-0.0003	-0.0023	0.5743	0.0046
			MSE	0.00089	0.00158	0.00792	0.00098
		$\hat{\beta}$	Bias	-0.5878	-0.5971	-0.5826	-0.5762
			MSE	0.00733	0.00749	0.00717	0.00704
	100	$\hat{\alpha}$	Bias	0.0005	0.0013	0.5451	-0.0014
			MSE	0.0002	0.00033	0.00327	0.0002
		$\hat{\beta}$	Bias	-0.5506	-0.5526	-0.5503	-0.5464
			MSE	0.003144	0.003149	0.003122	0.003082

Table 3: Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$: outlier in y.

σ_ϵ^2	n	Estimate	Statistic	Estimation Methods			
				Two group	Three group	Repetitive r = 3	Repetitive r = 4
16	10	$\hat{\alpha}$	Bias	-0.0022	0.0104	0.4447	-0.0035
			MSE	0.33157	0.56598	0.97914	0.64681
		$\hat{\beta}$	Bias	-0.5219	-0.5041	-0.5027	-0.5146
			MSE	0.3338	0.33399	0.42493	0.89781
	50	$\hat{\alpha}$	Bias	0.005	0.0045	0.4968	0.0042
			MSE	0.00273	0.00443	0.0119	0.00286
		$\hat{\beta}$	Bias	-0.5065	-0.5041	-0.5016	-0.5037
			MSE	0.00731	0.00701	0.00719	0.00707
	100	$\hat{\alpha}$	Bias	0.0014	0.0008	0.4898	-0.0026
			MSE	0.00041	0.00061	0.00333	0.00042
		$\hat{\beta}$	Bias	-0.5008	-0.4995	-0.4979	-0.4961
			MSE	0.002831	0.002764	0.002768	0.002731

Table 4: Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$: outlier in both (x, y).

$(\sigma_\delta^2, \sigma_\epsilon^2)$	n	Estimate	Statistic	Estimation Methods			
				Two group	Three group	Repetitive r = 3	Repetitive r = 4
$(16, 16)$	10	$\hat{\alpha}$	Bias	-0.0159	-0.0106	0.6588	-0.0131
			MSE	0.12335	0.25809	0.36305	0.41058
		$\hat{\beta}$	Bias	-0.7459	-0.7684	-0.715	-0.6691
			MSE	0.1585	0.19088	0.17847	0.26748
	50	$\hat{\alpha}$	Bias	0.0055	0.0037	0.5793	-0.0009
			MSE	0.00239	0.00485	0.01102	0.00264
		$\hat{\beta}$	Bias	-0.5885	-0.5959	-0.5911	-0.5748
			MSE	0.00856	0.00893	0.00861	0.00819
	100	$\hat{\alpha}$	Bias	0.001	0.0045	0.5447	0.0034
			MSE	0.00041	0.00077	0.00366	0.00041
		$\hat{\beta}$	Bias	-0.5509	-0.5526	-0.5472	-0.5476
			MSE	0.003312	0.003345	0.003266	0.003257

5. REAL DATA ANALYSIS

The real data analysis in this article seeks to determine the impact of GDP on HDI in Jordanwithin the period (1990-2017). The trend of both variables within the study period are given in Figure.1 and Figure 2.

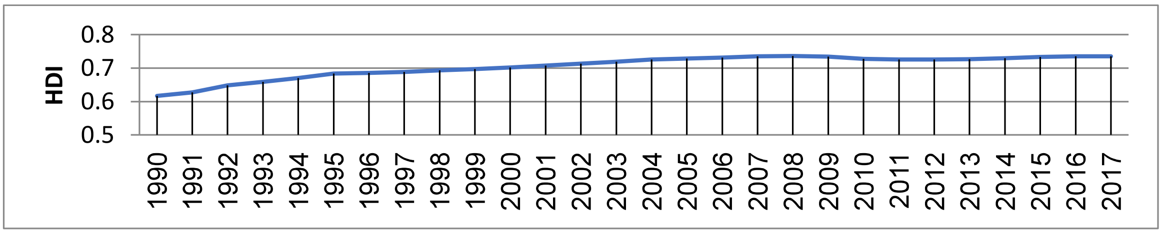

Figure.1 The trend of the Jordanian HDI within 1990-2017

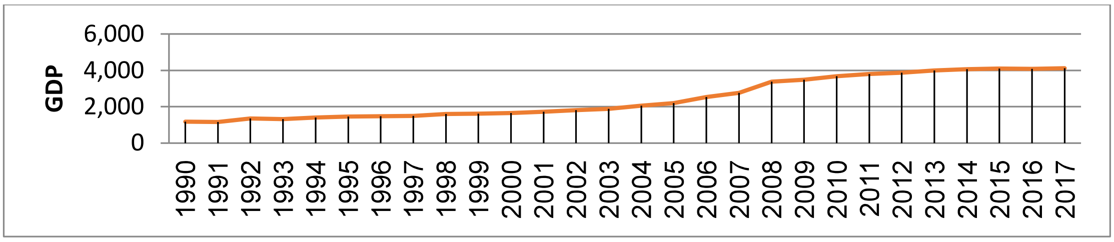

Figure.2 The trend of the National GDP within 1990-2017

Moreover, Table 5. representsthe descriptive statistics of both variables in general. It is worth to say that there is a strong positive significant correlation ($r = 0.761$, $p < 0.001$) between GDP and HDI in Jordan.

Table 5: Descriptive Statistics

Variable	Min	Max	Mean	STDEV	Correlation	P.
GDP	1158	4130	2476.82	1120.867	0.761	< 0.001
HDI	.617	.736	.70514	.034164		

Moreover, the scatter plot (Figure 3) suggest the type of the relationships to be (almost linear).

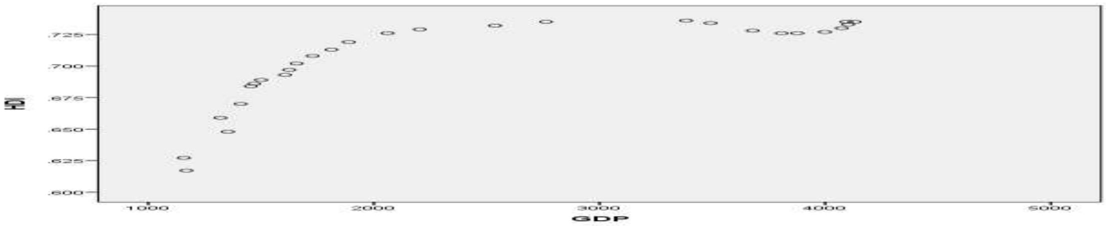

Figure.3 The scatter plot of HDI and GDP

Therefore, GDP and HDI can be modeled as a linear relationship, however, we believe that both variables are measured subject to error since the final value for each of these variables depends on several sub-factor. Hence, the MEM is the best model to be used to study the relationship between HDI and GDP which can be rewritten as

$$HDI = \alpha + \beta \times (GDP - \delta) + \epsilon.$$

Accordingly, Table 6, shows the results of all estimation methods considered in this article. The results indicated the proposed method with r =3 and the three-group methods gave more accurate estimators than the other estimation methods as can be seen in Figure 4.

Table 6: Parameter Estimation of HDI vs GDP

Method	criterion	$\hat{\beta}$	$\hat{\alpha}$
Classical	Two-Group	3.76E+4	-2.40E+4
	Three-Group	2.92E+4	-1.81E+4
Proposed	r = 3	6.11E+4	-4.06E+4
	r = 4	7.13E+4	-4.78E+4

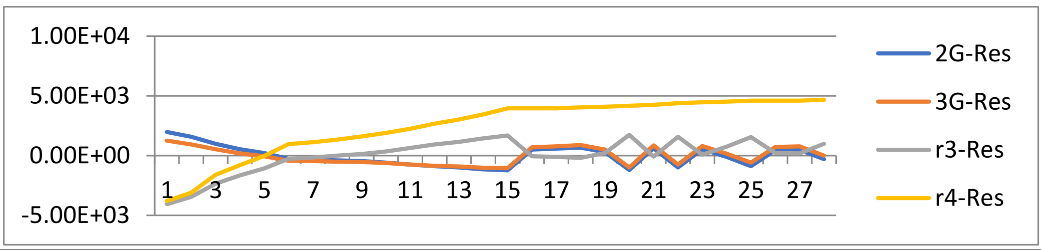

Figure 4 Residual Comparisons of the estimation methods

CONCLUDING REMARKS

This study proposed a new non-parametric estimation procedure to fit the structural MEM. The new procedure used repetitive Wald type estimation method. The Monte Carlo simulations provide a good evidence for the superiority of the proposed estimation procedure on the classical methods in cases of the data moderate or large sample size. Moreover, the estimation procedure applied on a real data to study the effect of the GDP on the HDI. The data analysis suggested that there is a strong positive relationship between both variables. Future work will be about finding the optimal *r* value in the proposed procedure.

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FRACTIONAL INTEGRAL FORMULAS INVOLVING (P-K)-MITTAG-LEFFER FUNCTION

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ABSTRACT

The objective of the paper is to introduce certain fractional integral formulas of (p-k)-Mittag-Leffer Function by using the generalized fractional integral operators (the Marchichev-Saigo-Maeda operators). Further integral formulas are also obtained involving Saigo and Riemann-Lioville integral operators as their special cases.

Keywords: (p-k) Pochhemmer symbol; Fractional Kinetic Equation; (p-k)-Mittag-Leffer Function; Laplace Transform.

1. INTRODUCTION AND PRELIMINARIES

Gehlot in [1] presented the following two parameter Pochhammer symbol defined as:

Definition 1. *Let* $w \in \mathbb{C}; p,k \in \mathbb{R}^+ - 0; n \in \mathbb{N}; \Re(w) > 0$, *then (p-k) Pochhammer symbol is defined as:*

$$_p(w)_{n,k} = \left(\frac{wp}{k}\right)\left(\frac{wp}{k}+p\right)\left(\frac{wp}{k}+2p\right)\cdots\left(\frac{wp}{k}+(n-1)p\right) = \frac{_p\Gamma_k(w+nk)}{_p\Gamma_k(w)}. \quad (1.1)$$

Gehlot in [1] introduced the two parameter gamma function defined as:

Definition 2.*Let* $w \in \mathbb{C} \setminus k\mathbb{Z}^-; p,k \in \mathbb{R}^+ - 0; n \in \mathbb{N}; \Re(w) > 0$, *then (p-k) Gamma function is defined as:*

$$_p\Gamma_k(w) = \int_0^\infty e^{-\frac{t^k}{p}} t^{w-1} dt \quad (1.2)$$

Recently in [2], Gehlot introduced the (p-k) Mittag-Leffler function defined as:

Definition 3.*Let* $p,k \in \mathbb{R}^+ - 0; \xi,\zeta,\tau \in \mathbb{C} \setminus k\mathbb{Z}^-; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\tau) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$, *then (p-k) Mitag-Leffler function is defined as:*

$$_pE^{\tau,q}_{k,\xi,\zeta}(z) = \sum_0^\infty . \frac{_p(\tau)_{nq,k}}{_p\Gamma_k(n\xi+\zeta)} \frac{z^n}{n!} \quad (1.3)$$

where $_p(\tau)_{nq,k}$ is two parameter Pochhammer symbol defined in equation (1.1).Following lemmas are required for our present study as follows:

Lemma 1.*For the (p-k) Pochhammer symbol and the* k *-Pochhammer symbol and the classical Pchhammer symbol it has*

$$ {}_p(w)_{n,k} = \left(\frac{p}{k}\right)^n (w)_{n,k} = p^n \left(\frac{w}{k}\right)_n . \qquad (1.4) $$

Lemma 2. *For the (p-k) Gamma function, the k -Gamma function and the classical Gamma function it has [1]*

$$ {}_p\Gamma_k(w) = \left(\frac{p}{k}\right)^{\frac{w}{k}} \Gamma_k(w) = \frac{p^{\frac{w}{k}}}{k} \Gamma\left(\frac{w}{k}\right). \; (1.5) $$

2. THE GENERALIZED FRACTIONAL INTEGRAL OPERATORS

The generalized fractional calculus operators (the Marchichev-Saigo-Maeda operators), involving the Appell's function or the Horn's $F_3()$ function in the kernel are defined as (see for details, Marichev [5], [9, 10, 11], Saigo and Maeda [8]).

Definition 4. *Let* $\delta,\delta',\nu,\nu',\eta \in \mathbb{C}$ *and* $x>0$, *then for* $\Re(\eta)>0$, *then*

$$ \left(I_{0,x}^{\delta,\delta',\nu,\nu',\eta} f\right)(x) = \frac{x^{-\delta}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\delta'} F_3\left(\delta,\delta',\nu,\nu';\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt \; (2.1) $$

and

$$ \left(I_{x,\infty}^{\delta,\delta',\nu,\nu',\eta} f\right)(x) = \frac{x^{-\delta'}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\delta} F_3\left(\delta,\delta',\nu,\nu';\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt, \; (2.2) $$

provided the integrals in equation (2.1) and (2.3) exist.
In equation (2.1) and (2.3), $F_3(.)$ denotes Appell's hypergeometric function [16] in two variables defined as:

$$ F_3(\delta,\delta',\nu,\nu';\eta;x,y) = \sum_{m,n=0}^{\infty} (\frac{(\delta)_m (\delta')_n (\nu)_m (\nu')_n}{(\eta)_{m+n}} \frac{x^m}{m!} \frac{x^n}{n!} \; \max\{|x|,|y|\} < 1). \; (2.3) $$

The above fractional integral operators in equation (2.1) and (2.3) can be written as follows:

$$ \left(I_{0,x}^{\delta,\delta',\nu,\nu',\eta} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{\delta,\delta',\nu+k,\nu',\eta+k} f\right)(x) \qquad (2.4) $$
$$ \left(\Re(\eta) \le 0; k = \left[-\Re(\eta)+1\right]\right) $$

and

$$\left(I_{x,\infty}^{\delta,\delta',\nu,\nu',\eta} f\right)(x)=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{\delta,\delta',\nu,\nu'+k,\eta+k} f\right)(x) \tag{2.5}$$
$$\left(\Re(\eta)\leq 0; k=\left[-\Re(\eta)+1\right]\right).$$

The following formulas are required for our present study as given in the following lemma [8, 9, 13].

Lemma 3. *Let* $\delta,\delta',\nu,\nu',\eta$ *and* $\rho\in\mathbb{C}, x>0$ *be such that* $\Re(\eta)>0$, *then*

$$\left(I_{0,x}^{\delta,\delta',\nu,\nu',\eta} t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\delta-\delta'-\nu)\Gamma(\rho+\nu'-\delta')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\delta-\delta')\Gamma(\rho+\eta-\delta'-\nu)}x^{\rho+\eta-\delta-\delta'-1} \tag{2.6}$$
$$(\Re(\rho)>\max\{0,\Re(\delta+\delta'+\nu-\eta),\Re(\delta'-\nu')\})$$

and

$$\left(I_{x,\infty}^{\delta,\delta',\nu,\nu',\eta} t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\delta+\delta')\Gamma(1-\rho-\eta+\delta+\nu')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\delta+\delta'+\nu')\Gamma(1-\rho+\delta-\nu)}x^{\rho+\eta-\delta-\delta'-1}$$
$$(\Re(\rho)<1+\min\{\Re(-\nu),\Re(\delta+\delta'-\eta),\Re(\delta+\nu'-\eta)\}).$$
(2.7)

The Hadamard product (or the convolution) of two analytic functions is very useful in the present work. Let

$$\phi(z)=\sum_{n=0}^{\infty} a_n z^n \quad (|z|<R_{\phi}) \tag{2.8}$$

and

$$\psi(z)=\sum_{n=0}^{\infty} b_n z^n \quad (|z|<R_{\psi}) \tag{2.9}$$

be two power series. Then, their Hadamard product is the power series defined by

$$(\phi*\psi)(z)=\sum_{n=0}^{\infty} a_n b_n z^n=(\psi*\phi)(z) \quad (|z|<R) \tag{2.10}$$

where

$$R=\lim_{n\to\infty}\left|\frac{a_n b_n}{a_{n+1}b_{n+1}}\right|=\left(\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\right).\left(\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|\right)=R_{\phi}.R_{\psi}, \tag{2.11}$$

thus, we have $R\geq R_{\phi}\,R_{\psi}$ [4, 7] (see also [15, 14] and the references cited therein).

Fox-Wright function ${}_p\Psi_q(z)\,(p,q\in\mathbb{N}_0)$ with p numerator and q denominator parameters defined for $a_1,\ldots,a_p\in\mathbb{C}$ and $b_1,\ldots,b_q\in\mathbb{C}\setminus\mathbb{Z}_0^-$ by (see [3, 6, 12, 16])

$$ {}_p\Psi_q \left[\begin{matrix} (a_1,\alpha_1),...,(a_p,\alpha_p); \\ (b_1,\beta_1),...,(b_q,\beta_q); \end{matrix} \quad z \right] = \sum_{n=0}^{\infty}{}' \frac{\Gamma(a_1+\alpha_1 n)...\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n)...\Gamma(b_q+\beta_q n)} \frac{z^n}{n!} \quad (2.12) $$

where the coefficients $\alpha_1,...,\alpha_p$, $\beta_1,...,\beta_q \in \mathbb{R}^+$ are such that

$$ 1+\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i \geq \quad (2.13) $$

3. FRACTIONAL INTEGRATION OF (P-K)- MITTAG-LEFFLER FUNCTION

In this section, we present certain fractional integral formulas involving (p-k)-Mittag-Leffler function ${}_pE^{\tau,q}_{k,\xi,\zeta}(z)$ by using the generalized fractional integral operators (the Marchichev-Saigo-Maeda operators).

Theorem 1. *Let* $x>0,\delta,\delta',\nu,\nu',\eta,\rho\in\mathbb{C}$ *and* $p,k\in\mathbb{R}^+-0;\xi,\zeta,\tau\in\mathbb{C}\backslash k\mathbb{Z}^-;\Re(\xi)>0,\Re(\zeta)>0,\Re(\tau)>0$ *and* $q\in(0,1)\cup\mathbb{N}$ *be such that* $\Re(\eta)>0$ *and* $\Re(\rho+\omega n)>max\{0,\Re(\delta+\delta'+\nu-\eta),\Re(\delta'-\nu')\}$ *then the following fractional integral formula holds true:*

$$ \left(I^{\delta,\delta',\nu,\nu',\eta}_{0,x}\left\{t^{\rho-1}{}_pE^{\tau,q}_{k,\xi,\zeta}\left(t^{\omega}\right)\right\}\right)(x) = x^{\rho+\eta-\delta-\delta'-1} 4mm \times_p E^{\tau,q}_{k,\xi,\zeta}\left(x^{\omega}\right) *_3\Psi_3 $$
$$ \left[\begin{matrix} (\rho,\omega),(\rho+\eta-\delta-\delta'-\nu,\omega),(\rho+\nu'-\delta',\omega); \\ (\rho+\nu',\omega),(\rho+\eta-\delta-\delta',\omega),(\rho+\eta-\delta'-\nu,\omega); \end{matrix} \quad x^{\omega} \right]. \quad (3.1) $$

Proof. Denote the left hand side of equation (3.1) by $\mathcal{I}$ .Then using the definition (1.3) and interchanging the order of integration and summation, we have

$$ \mathcal{I} = \sum_{0}^{\infty}{}' \frac{{}_p(\tau)_{nq,k}}{{}_p\Gamma_k(n\xi+\zeta)} \frac{1}{n!} \left(I^{\delta,\delta',\nu,\nu',\eta}_{0,x} t^{\rho+\omega n-1}\right)(x) \quad (3.2) $$

applying the result (2.11), equation (3.2) reduces to

$$ \mathcal{I} = \sum_{0}^{\infty} \frac{{}_p(\tau)_{nq,k}}{{}_p\Gamma_k(n\xi+\zeta)} \frac{x^{\rho+\omega n+\eta-\delta-\delta'-1}}{n!} \times $$
$$ \frac{\Gamma(\rho+\omega n)\Gamma(\rho+\omega n+\eta-\delta-\delta'-\nu)\Gamma(\rho+\omega n+\nu'-\delta')}{\Gamma(\rho+\omega n+\nu')\Gamma(\rho+\omega n+\eta-\delta-\delta')\Gamma(\rho+\omega n+\eta-\delta'-\nu)}, \quad (3.3) $$

after little simplification, the above equation (3.3) reduces to

$$\mathcal{I} = x^{\rho+\eta-\delta-\delta'-1}\sum_{0}^{\infty} \frac{{}_p(\tau)_{nq,k}}{{}_p\Gamma_k(n\xi+\zeta)}\times$$
$$\frac{\Gamma(\rho+\omega n)\Gamma(\rho+\eta-\delta-\delta'-\nu+\omega n)\Gamma(\rho+\nu'-\delta'+\omega n)}{\Gamma(\rho+\nu'+\omega n)\Gamma(\rho+\eta-\delta-\delta'+\omega n)\Gamma(\rho+\eta-\delta'-\nu+\omega n)}\frac{x^{\omega n}}{n!}. \quad (3.4)$$

Using equation (2.19), in view of (1.3) and (2.21), equation (3.4) gives the required result (3.1).

Theorem 2 *Let* $x>0,\delta,\delta',\nu,\nu',\eta,\rho\in\mathbb{C}$ *and* $p,k\in\mathbb{R}^{+}-0;\xi,\zeta,\tau\in\mathbb{C}\setminus k\mathbb{Z}^{-};$ $\Re(\xi)>0,\ \Re(\zeta)>0,\Re(\tau)>0$ *and* $q\in(0,1)\cup\mathbb{N}$ *be such that* $\Re(\eta)>0$ *and* $\Re(\rho-\omega n)<1+min\{\Re(-\nu),\Re(\delta+\delta'-\eta),\Re(\delta+\nu'-\eta)\}$ *then the following fractional integral formula holds true:*

$$\left(I_{x,\infty}^{\delta,\delta',\nu,\nu',\eta}\left\{t^{\rho-1}{}_pE_{k,\xi,\zeta}^{\tau,q}\left(\frac{1}{t^{\omega}}\right)\right\}\right)(x) = x^{\rho+\eta-\delta-\delta'-1}{}_pE_{k,\xi,\zeta}^{\tau,q}\left(\frac{1}{x^{\omega}}\right)*$$
$${}_3\Psi_3\left[\begin{matrix}(1-\rho-\nu,\omega),(1-\rho-\eta+\delta+\delta',\omega),(1-\rho-\eta+\delta+\nu',\omega); \\ (1-\rho,\omega),(1-\rho-\eta+\delta+\delta'+\nu',\omega),(1-\rho+\delta-\nu,\omega);\end{matrix}\ \frac{1}{x^{\omega}}\right]. \quad (3.5)$$

Proof. Proof of Theorem 2 is similar to that of Theorem 1.

3.1 Special Cases

Here we present some special cases by choosing suitable values of the parameters δ, δ', ν, ν' and η. If we put $\delta=\delta+\nu,\delta'=\nu'=0,\nu=-\eta,\eta=\delta$ in Theorems 1 and 2, we get certain interesting results concerning the Saigo fractional integral operators given by the following corollaries.

Corollary 1 *Let* $x>0,\delta,\nu,\eta,\rho\in\mathbb{C}$ *and* $p,k\in\mathbb{R}^{+}-0;\xi,\zeta,\tau\in\mathbb{C}\setminus k\mathbb{Z}^{-};\Re(\xi)>0,$ $\Re(\zeta)>0,\Re(\tau)>0$ *and* $q\in(0,1)\cup\mathbb{N}$ *be such that* $\Re(\delta)>0$ *and* $\Re(\rho+\omega n)>max\{0,\Re(\nu-\eta)\}$ *then the following fractional integral formula holds true:*

$$\left(I_{0,x}^{\delta,\nu,\eta}\left\{t^{\rho-1}{}_pE_{k,\xi,\zeta}^{\tau,q}\left(t^{\omega}\right)\right\}\right)(x) = x^{\rho-\nu-1}{}_pE_{k,\xi,\zeta}^{\tau,q}\left(x^{\omega}\right)*{}_2\Psi_2\left[\begin{matrix}(\rho,\omega),(\rho+\eta-\nu,\omega); \\ (\rho-\nu,\omega),(\rho+\eta+\delta,\omega);\end{matrix}\ x^{\omega}\right].$$
(3.6)

Corollary 2 *Let* $x>0,\delta,\nu,\eta,\rho\in\mathbb{C}$ *and* $p,k\in\mathbb{R}^{+}-0;\xi,\zeta,\tau\in\mathbb{C}\setminus k\mathbb{Z}^{-};\Re(\xi)>0,$ $\Re(\zeta)>0,\Re(\tau)>0$ *and* $q\in(0,1)\cup\mathbb{N}$ *be such that* $(\delta)>$ *and* $\Re(\rho-\omega n)<1+min\{\Re(\nu),\Re(\eta)\}$ *then the following fractional integral formula holds true:*

$$\left(I_{x,\infty}^{\delta,\nu,\eta}\left\{t^{\rho-1}{}_{p}E_{k,\xi,\zeta}^{\tau,q}\left(\frac{1}{t^{\omega}}\right)\right\}\right)(x)=x^{\rho-\nu-1}{}_{p}E_{k,\xi,\zeta}^{\tau,q}\left(\frac{1}{x^{\omega}}\right)*{}_{2}\Psi_{2}\left[\begin{matrix}(1-\rho+\nu,\omega),(1-\rho+\eta,\omega); & \frac{1}{x^{\omega}}\\ (1-\rho,\omega),(1-\rho+\delta+\nu+\eta,\omega); & \end{matrix}\right].$$

(3.7)

CONCLUSION

All the finding in this paper are general in nature. Various results as special cases can be easily obtained by employing the particular values to the parameters involving in our findings.

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THE DYNAMICAL BEHAVIOR OF STAGE STRUCTURED PREY-PREDATOR MODEL IN THE PRESENCE OF HARVESTING AND TOXIN

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ABSTRACT

In this paper, a mathematical model, consists from a prey-predator system with stagestructure in the presence of harvesting and toxicity has been proposed and studied by using the classicLotka-Volterra type of functional response. The existence, uniqueness and boundedness of the solution of the proposed model are discussed. The existence and the stability analyses of all possible equilibrium points are studied. The global stability of these equilibrium points are performed with suitableLyapunov functions. Finally, numerical simulations are carried out not only to confirm the theoreticalresults obtained, but alsoto show the effects of variation of each parameter on the proposed model.

Keywords: Prey-predator, functional response, stability analysis, Lyapunove function.**1**

1. INTRODUCTION

The prey-predator system is one of the most important topics in the ecosystem. It is used to solved many complex problems or which cannot be predicted with on the ground and thus is considered an alternative method in improving our knowledge of the physical and biologicalprocessesrelated to the environment.One of the most serious problems that threaten the ecosystem is over-harvesting of living things, because of the massive population increase and the desire of people to get more resources, that led to the danger to the ecosystem and has become a problem that worries alot. Several models were proposed according to harvest models [3,6,7,18,23]. While many researchers have tried to limit this problem by suggesting a model containing a refuge to save prey from extinction due to over-harvesting, and predation for example [1,13,24]. On the other hand, the age factor has a significant impact on the rate of growth and reproduction, in recent years, many prey-predator models based on age-structureare studied byauthors [4,8,15]. The other major problem affecting the ecosystem is pollution caused by toxic substances, many studies have considered on the environmental effects of toxic substances, Hallam and Clark [22] they studied the effects of toxic substances on exposed populations. In addition, Hallam and De Luna [21] have discussed the effects of a toxin through the food chain of the population. While Friedman and Shukla [10] developed the Models of predator-prey systems in a polluted closed environment with singlespecies. Chattopadhyay [12] studied the effects of toxic substances on two competing species and noted that the toxic substances have some stabilizing effect on keep the system.Mortoja et al. [17] considered two types of factors such as anti-predator behavior and group defense of stage-structure model. There is no doubt that the presence of toxicity will affect the harvest, some studies that focused on the existence of harvest and toxic substance [5,9,11,14,16,19,20]. Finally,Majeed [2] suggests model contains stage structures in both populations with the effect of toxicant. In this paper, the stage-structured of prey-predator model with harvesting and toxicity has been proposed and studied. The considered model consists of four nonlinear ordinary differential equations to describethe interactions by using Lotka-Volterra type of functional response.This system is analyzed by using the linear stability analyses to find the conditionsfor which the feasible equilibrium points are stable. Global stability conditions for proposed model are described by using appropriate Lyapunove functions.

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2. MODEL FORMULATION

In this section, the model consists of two species prey andpredator, each species divided into two classes: one is immature and other is mature, which are denoted to their population's sizes at time Tby X(T), Y(T), Z(T)and W(T) respectively. Now, in order to formulate the dynamics of such system, the following assumptions are considered:

The immature of prey and predator grown up to be mature with grown up ratesη_1 and$\eta_2$respectively. The immature prey depends completely in its feeding on mature prey that growth logistically with an intrinsic growth rate r and carrying capacity$k > 0$ in absence ofmaturepredator. Also the immature predator depends completely in its feeding on mature predator thatconsumes the immature andmature prey with the classicalLotka-Volterra functional response with consumption rates θ_1 andθ_2, respectively, therefore the predatorspecies growth due to attack by mature predator on immature and mature prey with conversion rates $0 < e_1 < 1$and $0 < e_2 < 1$. However, in absence of prey species the predatorspeciesdecay exponentially with the mortalityrates $\gamma_1$and$\gamma_2$of immatureand mature predator respectively. Moreover, the immaturepredator can't attack any of the preys, rather than that it depends completely on his parents, so that it feeds on the portion of up taken food by maturepredator from the first andsecond preys with portion rates $0 < n_1 < 1$ and $0 < n_2 < 1$respectively. Finally, $\varphi_i$andδ_i i=1,2,3,4 arethe catchabilitycoefficients and the toxicity coefficients of prey species andpredator species respectively. According above assumptions, the model is formulated as follows:

$$\begin{aligned}
\frac{dX}{dT} &= rY\left(1-\frac{Y}{K}\right) - \eta_1 X - \delta_1 X^2 - \varphi_1 X - \theta_1 XW \\
\frac{dY}{dT} &= \eta_1 X - \delta_2 Y^2 - \varphi_2 Y - \theta_2 YW \\
\frac{dZ}{dT} &= n_1 e_1 \theta_1 XW + n_2 e_2 \theta_2 YW - \eta_2 Z - \delta_3 Z - \varphi_3 Z - \gamma_1 Z \\
\frac{dW}{dT} &= \eta_2 Z + (1-n_1) e_1 \theta_1 XW + (1-n_2) e_2 \theta_2 YW - \delta_4 W - \varphi_4 W - \gamma_2 W
\end{aligned} \tag{1}$$

In order to simplify the system, the number of parameters is reduced by using the following dimensionlessvariables and parameters:

$$t = rT,\ x = \frac{X}{K},\ y = \frac{Y}{k},\ z = \frac{Z}{K},\ w = \frac{W}{k},\ \alpha_i = \frac{\eta_i}{r},\ \beta_i = \frac{\theta_i k}{r},\ d_i = \frac{\gamma_i}{r},\ h_j = \frac{\varphi_j}{r},$$

$$u_j = \frac{\delta_j k}{r}, \beta_{i+2} = \frac{n_i e_i \theta_i k}{r},\ \beta_{i+4} = \frac{(1-n_i) e_i \theta_i k}{r},\ \text{where } i = 1,2 \text{ and } j = 1,2,3,4$$

Then dimensional system (1) becomes:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{y(1-y)}{x} - (\alpha_1 + h_1) - u_1 x - \beta_1 w\right] = x f_1(x,y,z,w) \\
\frac{dy}{dt} &= y\left[\frac{\alpha_1 x}{y} - u_2 y - h_2 - \beta_2 w\right] = y f_2(x,y,z,w) = y f_2(x,y,z,w) \\
\frac{dz}{dt} &= z\left[\frac{\beta_3 xw}{z} + \frac{\beta_4 yw}{z} - (\alpha_2 + u_3 + h_3 + d_1)\right] = z f_3(x,y,z,w) \\
\frac{dw}{dt} &= w\left[\frac{\alpha_2 z}{w} + \beta_5 x + \beta_6 y - (u_4 + h_4 + d_2)\right] = w f_4(x,y,z,w)
\end{aligned} \tag{2}$$

Obviously the interaction functions of the system (2) arecontinuous and havecontinuous partialderivatives on the following positive four dimensional space:

$$R_+^4 = \{(x,y,z,w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}.$$

Therefore, these functions are Lipschitzian on R_+^4 , and hence the existence anduniqueness of the solution for system(2). Further, all the solutions of system (2) with non-negative initialconditions are uniformly bounded as shown in the followingtheorem.

Theorem 1: *All the solutions of system* (2)*are uniformly bounded.*

Proof. let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2) with$(x_0, y_0, z_0, w_0) \in R_+^4$. Nowconsider a function: $V(t) = x(t) + y(t) + z(t) + w(t)$, and then take the time derivative offunction: $V(t)$ alonge the solution of the system (2), So, due to the fact that theconversion rate constant from immature andmature prey population tomature and immature predator population cannot exceeding the maximumpredationrate constant from mature predator population to immature and mature prey population, hence from the biological point of view, always $\beta_1 > \beta_3 + \beta_5$ and $\beta_2 > \beta_4 + \beta_6$, we get:

So, $\frac{dV}{dt} + SV \leq \frac{1}{4}$, where $S = min\{h_1, h_2, (u_3 + h_3 + d_1), (u_4 + h_4 + d_2)\}$.

Now by solving this differential inequality for the initial value $V(0) = V_0$, we get:

$0 \leq V(t) \leq \frac{1}{4S}$, $\forall\, t > 0$. Hence all the solutions of system (2) are uniformlybounded .□

3. THE EXISTENCE OF EQUILIBRIUM POINTS

In this section, the existence of all possible equilibrium points of system (2)is discussed. It is observed that, system (2) has at most three nonnegative equilibrium pointswhich are in thefollowing:

- The equilibrium point $E_0 = (0, 0, 0, 0)$ always exists.
- The equilibrium point $E_1 = (\tilde{x}, \tilde{y}, 0, 0)$, exists uniquely in Int.R_+^2 if the following condition hold:

$$\alpha_1 + h_1 < \frac{\alpha_1}{h_2}. \quad (3)$$

- Finally the positiveequilibrium point $E_2 = (\overset{*}{x}, \overset{*}{y}, \overset{*}{z}, \overset{*}{w})$, exists if the following condition hold:

$$\overset{*}{x} > \frac{\overset{*}{y}\left(u_2 \overset{*}{y} + h_2\right)}{\alpha_1}. \quad (4)$$

4. THE STABILITY ANALYSIS

In this section the local stabilityanalysis of system (2) around each of the above equilibrium points is discussed through computing the Jacobianmatrix$J(x, y, z, w)$ of system (2):

- The characteristicpolynomial of the Jacobian matrix of system(2)at E_0, $J_0 = J(E_0)$gives the four eigenvalues of J_0 with negative real parts provided that the following condition holds:

$$h_2 > 1. \quad (5)$$

  Then E_0 is locally asymptotically stable in R_+^4, under the condition (5) . However, it is a saddle point (unstable) otherwise.

- The characteristicpolynomial of the Jacobian matrix of system(2)at E_1,$J_1 = J(E_1)$ gives the four eigenvalues ofJ_1 with negative real parts due to the following conditions:

$$\tilde{y} > \frac{1}{2}. \quad (6)$$

$$(u_4 + h_4 + d_2) > (\beta_5 \tilde{x} + \beta_6 \tilde{y}). \quad (7)$$

$$(\alpha_2 + u_3 + h_3 + d_1)\big((u_4 + h_4 + d_2) - (\beta_5 \tilde{x} + \beta_6 \tilde{y})\big) > \alpha_2(\beta_3 \tilde{x} + \beta_4 \tilde{y}). \quad (8)$$

  Hence, E_1 is locally asymptotically stable in R_+^4under the conditions(6-8). However, it is a saddle (unstable) point otherwise.

- Finally, then the characteristicequation of the Jacobian matrix of system(2)at E_2,J_2 is given by:

$$[\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4] = 0, \quad \text{where} \quad (9)$$

$$A_1 = -(c_{11} + c_{22} + c_{33} + c_{44}) > 0.$$

$$A_2 = c_{33}c_{44} - c_{34}c_{43} + (c_{11} + c_{22}) + (c_{33} + c_{44}) + c_{11}c_{22} - c_{12}c_{21} - c_{24}c_{42} + c_{14}c_{41}.$$

$$A_3 = -(c_{11} + c_{22})(c_{33}c_{44} - c_{34}c_{43}) - (c_{33} + c_{44})(c_{11}c_{22} - c_{12}c_{21}) + (c_{11} + c_{33})(c_{24}c_{42}) - (c_{12}c_{24}c_{41} + c_{21}c_{42}) + c_{14}c_{41}(c_{22} + c_{33}) - (c_{24}c_{32}c_{43} + c_{14}c_{31}c_{43}).$$

$$A_4 = (c_{33}c_{44} - c_{34}c_{43})(c_{11}c_{22} - c_{12}c_{21}) - c_{11}c_{24}c_{33}c_{42} + c_{33}(c_{12}c_{24}c_{41} + c_{21}c_{42}) - (c_{22}c_{33})(c_{14}c_{41}) + c_{11}(c_{24}c_{32}c_{43} + c_{14}c_{31}c_{43}) - c_{43}(c_{12}c_{24}c_{31} + c_{14}c_{21}c_{32}).$$

where, $c_{11} = -(\alpha_1 + h_1) - 2u_1\overset{*}{x} - \beta_1\overset{*}{w}$, $c_{12} = 1 - 2\overset{*}{y}$, $c_{13} = 0$, $c_{14} = -\beta_1\overset{*}{x}$,
$c_{21} = \alpha_1$, $c_{22} = -2u_2\overset{*}{y} - h_2 - \beta_2\overset{*}{w}$, $c_{23} = 0$, $c_{24} = -\beta_2\overset{*}{y}$, $c_{31} = \beta_3\overset{*}{w}$,

$c_{32} = \beta_4 \overset{*}{w}, c_{33} = -(\alpha_2 + u_3 + h_3 + d_1),\ c_{34} = \beta_3 \overset{*}{x} + \beta_4 \overset{*}{y},\ c_{41} = \beta_5 \overset{*}{w},$
$c_{42} = \beta_6 \overset{*}{w},\ c_{43} = \alpha_2,\ c_{44} = \beta_5 \overset{*}{x} + \beta_6 \overset{*}{y} - (u_4 + h_4 + d_2).$

Now by using Routh-Hawirtizcriterion equation (9) has roots (eigenvalues) with negative real parts if and only if $A_i > 0,\ i = 1,3,4$ and $\Delta = (A_1 A_2 - A_3)A_3 - A_1^2 A_4 > 0$. Clearly, $A_i > 0$ provided that:

$$\overset{*}{y} > \frac{1}{2}. \tag{10}$$

$$(u_4 + h_4 + d_2) > \beta_5 \overset{*}{x} + \beta_6 \overset{*}{y}. \tag{11}$$

$$(\alpha_2 + u_4 + h_4 + d_2)\left((u_4 + h_4 + d_2) - \beta_5 \overset{*}{x} + \beta_6 \overset{*}{y}.\right) > \alpha_2 \left(\beta_3 \overset{*}{x} + \beta_4 \overset{*}{y}\right). \tag{12}$$

$$\frac{\left(1 - 2\overset{*}{y}\right)\left(\beta_2 \overset{*}{y}\right)\beta_3 \overset{*}{w}}{\left(\beta_1 \overset{*}{x}\right)\left(\beta_4 \overset{*}{w}\right)} < \alpha_1 < \frac{\left(1 - 2\overset{*}{y}\right)\left(\beta_2 \overset{*}{y}\right)\beta_5 \overset{*}{w}}{\left(\beta_1 \overset{*}{x}\right)\left(\beta_6 \overset{*}{w}\right)} \tag{13}$$

Hence, Δ will be positive if in addition of conditions (10-14).Therefore, all the eigenvalues of J_2 have negative real parts under the given conditions and hence E_2 is locallyasymptotically stable. However, it is unstable otherwise.

5. GLOBAL STABILITY ANALYSIS

In this section the global stabilityanalysis for the equilibrium points which are locally asymptotically stable of system (2) is studied analytically with the help of Lyapunov method we get:

- Assume that $E_0 = (0,0,0,0)$ is locallyasymptoticallystable in R_+^4. Then E_0 is globally asymptoticallystable on the region $\omega_0 \subset R_+^4$, where $\omega_0 = \{(x, y, z, w) \in R_+^4 : y > 1\}$.
- Assume that $E_1 = (\tilde{x}, \tilde{y}, 0,0)$ is a locallyasymptoticallystable in R_+^4. Then E_1 is a globally asymptotically stable on the region $\omega_1 \subset R_+^4$, that satisfies the following conditions:

$$y > y^2. \tag{14}$$

$$\left(\frac{1 - (y - y^2)}{\tilde{x}} + \frac{\alpha_1}{\tilde{y}}\right) \le 2\sqrt{\left(u_1 + \frac{(y - y^2)}{x\tilde{x}}\right)\left(u_2 + \frac{\alpha_1 x}{y\tilde{y}}\right)} \tag{15}$$

- Assume that $E_2 = (\overset{*}{x}, \overset{*}{y}, \overset{*}{z}, \overset{*}{w})$ of system (2)is locally asymptoticallystable in the R_+^4. Then E_3 is a globallyasymptoticallystable on any region $\omega_2 \subset R_+^4$, that satisfies the following conditions:

$$y > y^2 \tag{16}$$

$$\left(\frac{1 - (y - y^2)}{\overset{*}{x}} + \frac{\alpha_1}{\overset{*}{y}}\right) \le \sqrt{\left(u_1 + \frac{(y - y^2)}{x\overset{*}{x}}\right)\left(u_2 + \frac{\alpha_1 x}{y\overset{*}{y}}\right)}. \tag{17}$$

$$(\beta_1 - \beta_5) \le \sqrt{\frac{1}{2}\left(u_1 + \frac{(y - y^2)}{x\overset{*}{x}}\right)\left(\frac{\alpha_2 z}{w\overset{*}{w}}\right)}. \tag{18}$$

$$(\beta_2 - \beta_6) \le \sqrt{\frac{1}{2}\left(u_2 + \frac{\alpha_1 x}{y\overset{*}{y}}\right)\left(\frac{\alpha_2 z}{w\overset{*}{w}}\right)}. \tag{19}$$

$$\frac{\alpha_2}{\overset{*}{w}} \le \sqrt{2\left(\frac{\beta_3 xw + \beta_4 yw}{z\overset{*}{z}}\right)\left(\frac{\alpha_2 z}{w\overset{*}{w}}\right)}. \tag{20}$$

6. NUMERICAL ANALYSIS OF SYSTEM

In this section, the dynamical behavior of system (2) is studied numerically for one set of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and

second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfiesstabilityconditions of the positiveequilibrium point, system (2) has a globally asymptotically
stable positive equilibrium point .

$$\alpha_i = 0.5, 0.2,\ u_j = h_j = d_i = 0.1,\ \beta_{j+2} = 0.3,\ \beta_i = 0.6\,,\ i = 1,2\ and\ j = 1,2,3,4 \qquad (21)$$

Further, withvarying one parameter each time, it isobserved that varying the parametersvalues,d_1 , $\alpha_2, u_i\,, h_i\,,\ i = 1{,}3and\beta_j\,,\ j = 1,2,3,4,5.$, do not have any effect on the dynamicalbehavior of system (2) and the solution of the system stillapproaches to positiveequilibrium point $E_2 = \left(\overset{*}{x}, \overset{*}{y}, \overset{*}{z}, \overset{*}{w}\right)$. By varying$\alpha_1$ in the range$0.001 \le \alpha_1 < 0.01$, causesextinction of all species and the solution of system (2) approaches asymptotically to E_0, as shown in Fig.(1)a, for typicalvalue$\alpha_1 = 0.005$, while the increasing of this parameterin the range$0.01 \le \alpha_1 < 0.023$ the solution of system (2) approachesasymptotically to $E_1 = (\tilde{x}\,, \tilde{y}, 0\,, 0\,)$ in the int. of R^4_+,as shown in Fig.(1) b , for typicalvalue$\alpha_1 = 0.02$, further increasing this parameter further in the range$0.023 \le \alpha_1 < 1$the solution of system (2)approachesasymptotically to the equilibrium point in the int. of R^4_+, as shown in Fig.(1)c, for typical value $\alpha_1 = 0.1$.

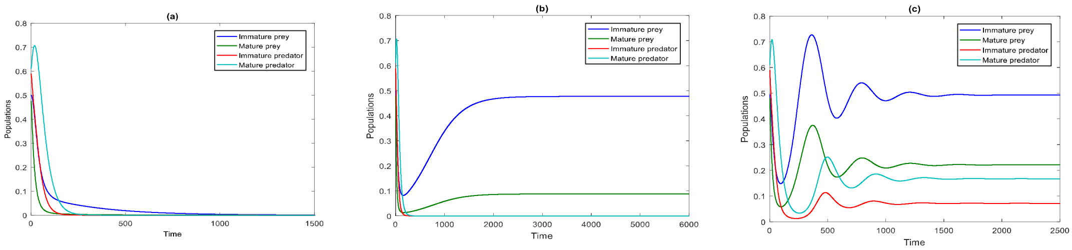

Fig. (1): (a) Timeseries of the solution of system (2) for the data given by (22) with $\alpha_1 = 0.005$, whichapproaches to$E_0 = (\,0\,,0\,,0\,,0\,)$, (b): Time series of thesolution of system (2) for the data given by (22)with$\alpha_1 = 0.02$, whichapproaches to $E_1 = (0.477,0.088,0,0)$, (c): Time series of the solution of system (2) for the data given by (21) with $\alpha_1 = 0.1$, whichapproaches to$E_2 = (0.493,0.222,0.072,0.168)$ in the int. of R^4_+.

Varying the parameterh_2 , and keeping the rest of parameters as data given in(21) in the range$0.01 \le h_2 < 0.484$, it observed that the solution of system (2)approachesasymptotically to E_2 . However, increasing this parameter in the range$0.484 \le h_2 < 0.85$causesextinction in the predatorspecies and the solution of system (2)approachesasymptotically to $E_1 = (\tilde{x}\,, \tilde{y}, 0\,, 0\,)$ in the int. of R^2_+,then increasing in the range$0.85 \le h_2 < 1$causesextinction in all species and the solution of system (2)approachesasymptotically to $E_0 = (\,0\,,0\,,0\,,0\,)$.The effect of Varying the parameteru_2 , with $0.01 \le u_2 < 1.226$and keeping the rest of parameters as data given in(21), it is observed that the solution of system(2)stillapproaches asymptotically to E_2, while the increasing of this parameter for $1.226 \le u_2 < 2$ leads that the solution of system (2)approachesasymptotically to E_1. Moreover, β_6,keeping the rest of parametersvalues as data given in (21) with $0.01 \le \beta_6 < 0.106$ the solution of system (2)approachesasymptoticallyE_1,while the increasing of this parameter for $0.106 \le \beta_6 \le 0.3$ leads that the solution of system (2)approachesasymptoticallyE_2. Finally, the parametersu_4, h_4 and d_2, have the same effect on the behavior of solution of system (2) and keeping the rest of parameters as data given in (21) in the range$0.01 \le u_4 < 0.251$, it is observed that the solution of system(2)stillapproachesasymptotically to E_2, while the increasing of this parameter for $0.251 \le u_4 < 1$ leads that the solution of system (2)approachesasymptotically to E_1.

CONCLUSIONS AND DISCUSSION

In this paper, we proposedand analyzed an ecological model that described the dynamical behavior of the stage-structured of prey-predator in both species with harvesting and toxicity. The modelincluded four non-linear autonomous differentialequations that describe the dynamics of four differentpopulation, namely first immatureprey (x), mature prey (y), immaturepredator(z) and maturepredator(w). The boundedness of system (2) has been discussed. The existenceconditions of all possible equilibrium points are obtain. The local as well as global stabilityanalyses of these points are carried out. Finally, numerical simulation is used to specify the control set of parameters that affect the dynamics of the system and confirm our obtained analyticalresults. Therefore system (2) has been solved numerically for different sets of initial points and a set of parametersstarting with the hypothetical set of data given by eq. (21) and the following observations are obtained. The system within the set of parametersimposed does not have a periodicsolution. For the sethypothetical parametersvalue given in (21), the system (2) approachesasymptotically to globally stablepositive point $E_2 = (0.292, 0.422, 0.146, 0.341)$. Further, withvarying one parameter each time, it is observed that varying the parametersvalues, d_1, α_2, u_i, h_i, $i = 1,3$andβ_j, $j = 1,2,3,4,5$. do not have any effect on the dynamical behavior of system (2) and the solution of the system still approaches$E_2 = \left(\overset{*}{x}, \overset{*}{y}, \overset{*}{z}, \overset{*}{w}\right)$. The parameters$\alpha_1$ and h_2 have a bifurcation with two values0.02, 0.1, 0.484 and 0.85 respectively. Finally, the parametersu_2, β_6, u_4, h_4 and $d_2$have a bifurcation with values 1.226, 0.106 and $u_4 = h_4 = d_2 =$ 0.251 respectively.

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EXPONENTIATED Q-EXPONENTIAL DISTRIBUTION

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ABSTRACT

In this paper, we investigate the properties of the exponentiated q-exponential distribution. The distribution has been compared with the q-exponential distribution in terms of the moment measures, distribution measures, survival function and failure rate function. Also, the maximum likelihood estimators of the unknown parameters in both distributions have been investigated. Finally, a real time to event data analysis is discussed.

Keywords: *Exponentiated Family, hazard function, Survival Analysis*.

1. INTRODUCTION

The q-Exponential distribution (QED) introduced in [9] by maximizing the Tsallis entropy with respect to a moment constraints. This proposal enables the development of statistical distributions used as an alternative to the classical exponential distribution in fitting growth or time to event data. Moreover, The QED is a generalization of some lifetime distribution such as Lomax distribution, and it is a particular case of the generalized type II Pareto distribution [2]. The QED probability density function $f(x)$ of some variable $X$is defined as [13]:

$$f(x,\lambda,q) = (2-q)\lambda e_q(-\lambda x); \quad where x \in \begin{cases} (0,\infty) for\ 1 \le q \\ \left[0, \frac{1}{\sqrt{\lambda(1-q)}}\right] for q < 1 \end{cases} \tag{1}$$

Where; $e_q(x) = \begin{cases} (1+(1-q)x)^{\frac{1}{1-q}}; & if \quad q \neq 1 \\ e^x \quad ; & if \quad q = 1 \end{cases}$ given that q < 2 and $\lambda > 0$.

Also, the cumulative distribution function cdf of QED is

$$F(x,\lambda,q) = 1 - [1 + (q-1)\lambda x]^{\frac{2-q}{1-q}} \tag{2}$$

Since the last few decades, generalized models are more useful in biostatistics and other fields such as medical, health, and reliability analysis. These generalizations include the idea of exponentiated distribution which introduced by [10] who discussed a new family of distributions termed as an exponentiated exponential distribution. [4] studied beta exponentiated Weibull distribution. [5] Discussed the exponentiated moment exponential distribution and generalized exponentiated moment exponential distribution among others.

2. EXPONENTIATED Q-EXPONENTIAL DISTRIBUTION

The idea of exponentiated distribution was introduced by [3]who discussed a new family of distributions they observed that many properties of the new family [8], and a number of authors have developed various category of these distributions, The Exponentiated Exponential distribution proposed by [3], however, [12] introduced the Exponentiated Weibull distribution and in a similar way, [14] proposed the exponentiated gamma and exponentiated Frechet and exponentiated Gumbel distributions [11].

The exponentiated exponential distribution is generalization of the standard exponential distribution,the family has two parameters (scale and shape), such an addition of parameters makes the resulting distribution richer and more flexible for modeling data, [7] added positive parameter to a general of survival function.

Assume that T is a continuous random variable with probability density function (pdf) g*(t)* and cumulative distribution function (cdf) *G(t)*, then the exponentiated *cdf* and *pdf* are defined respectively as [1]:

$$G_\alpha(t) = (F(t))^\alpha; \quad \alpha \geq 1 \qquad \text{And} \qquad g_\alpha(t) = \alpha f(t)(F(t))^{\alpha-1}$$

Accordingly, the *cdf* and *pdf* when $\boldsymbol{q \neq 1}$of the Exponentiated QED are given respectively as:

$$\mathbf{G_\alpha(t,\lambda,q) = \left(1 - [1 + (q-1)\lambda t]^{\frac{2-q}{1-q}}\right)^{\alpha}} \tag{3}$$

and

$$\boldsymbol{g_\alpha(t,\lambda,q) = \alpha(2-q)\lambda e_q(-\lambda t)\left(1 - [1 + (q-1)\lambda t]^{\frac{2-q}{1-q}}\right)^{\alpha-1}} \tag{4}$$

Where, $x > 0$, α, λ *and* q are all real positive number which α *and* q play the role of the shape and scale parameters [6].

3. RELIABILITY MEASURES:

Survival time is defined as the time from the fixed original point to the beginning of the event of interest. Assume for now that T is a continuous random variable with probability density function $(pdf)f(t)$and cumulative distribution function$(cdf)F(t)$giving the probability that the event has occurred by duration t, survival function $S(t)$indicates the probability that the event of interest has not yet occurred by time t is given by. The time to failure analysis deals with the length of time T that a system remains operational until experiencing a failure [15], then the hazard function is the ratio of the probability density function to survival function$\left\{h(t) = \frac{f(t)}{S(t)}\right\}$.

***Corollary* (1):** Let T be a r.v. from QED distribution given in Eq.(1) and Eq.(2) then the survival function and the failure rate function (Hazard function) are given respectively as:

$$S(t,\lambda,q) = [1+(q-1)\lambda t]^{\frac{2-q}{1-q}} \qquad \text{And} \qquad h(t,\lambda,q) = \frac{(2-q)\lambda e_q(-\lambda t)}{[1+(q-1)\lambda t]^{\frac{2-q}{1-q}}}$$

***Corollary* (2):** Let T be a r.v. from Exponentiated QED distribution given in Eq.(3) and Eq.(4) then:

$$\mathrm{S}_\alpha(\mathrm{t},\lambda,\mathrm{q}) = 1-\left(1-[1+(\mathrm{q}-1)\lambda \mathrm{t}]^{\frac{2-\mathrm{q}}{1-\mathrm{q}}}\right)^{\alpha}, \ \mathrm{h}_\alpha(\mathrm{t},\lambda,\mathrm{q}) = \frac{\alpha(2-\mathrm{q})\lambda \mathrm{e}_\mathrm{q}(-\lambda \mathrm{t})\left(1-[1+(\mathrm{q}-1)\lambda \mathrm{t}]^{\frac{2-\mathrm{q}}{1-\mathrm{q}}}\right)^{\alpha-1}}{1-\left(1-[1+(\mathrm{q}-1)\lambda \mathrm{t}]^{\frac{2-\mathrm{q}}{1-\mathrm{q}}}\right)^{\alpha}}.$$

Moment Measures

Therefore, we derived expressions for some important moment measures.

***Corollary* 3:** Let T be a r.v. from QED distribution given in Eq.(1) and Eq.(2) then the first four moments of the distribution when q > 1 are given in Table 1.

Moment	Mathematical expression
1	$\frac{1}{3\lambda - 2\lambda q}, q < \frac{3}{2}$
2	$\frac{2}{\lambda^2(6q^2-17q+12)}, q < \frac{4}{3}$

Table 1. The first two moments of QED

***Corollary* 4:** Let T be a r.v. from Exponentiated QED (EQED) distribution given in Eq.(3) and Eq.(4) then the first four moments of the distribution when q > 1 and $\alpha = 2$are given in Table2.

Moment	Mathematical expression
1	$\frac{2(2-q)\left(\frac{\Gamma\left(\frac{3-2q}{q-1}\right) - {}_2F_1\left(\frac{5-3q}{q-1},\frac{2-q}{q-1};\frac{3-2q}{q-1};1\right)\Gamma\left(\frac{5-3q}{q-1}\right)\Gamma\left(\frac{3q-4}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)} - \frac{{}_2F_1(2,\frac{1}{q-1};\frac{4q-5}{q-1};1)\Gamma(\frac{4-3q}{q-1})}{\Gamma(\frac{2-q}{q-1})}\right)}{\lambda(q-1)^2}$
2	$\frac{2(2-q)\left(\frac{2\,\Gamma\left(\frac{4-3q}{q-1}\right) - {}_2F_1\left(\frac{6-4q}{q-1},\frac{2-q}{q-1};\frac{4-3q}{q-1};1\right)\Gamma\left(\frac{6-4q}{q-1}\right)\Gamma\left(\frac{4q-5}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)} - \frac{2\ {}_2F_1(3,\frac{1}{q-1};\frac{5q-6}{q-1};1)\Gamma(\frac{5-4q}{q-1})}{\Gamma(\frac{2-q}{q-1})}\right)}{\lambda^2(q-1)^3}$

Table 2. The first two moments of EQED

Where: $2F1(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{z}) = \frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{b})\Gamma(\mathrm{c}-\mathrm{b})}\int_0^1 \mathrm{t}^{\mathrm{b}-1}(1-\mathrm{t})^{\mathrm{c}-\mathrm{b}-1}(1-\mathrm{zt})^{-\mathrm{a}}\mathrm{dt}$

Then, $\mathrm{E(X)} = \frac{1}{\lambda(3\mathrm{q}-5)}$ *and* $E(X^2) = \frac{9q-16}{\lambda^2(18\mathrm{q}^3-81\mathrm{q}^2+121\mathrm{q}-60)}$

4. MAXIMUM LIKELIHOOD ESTIMATION

Numerous estimation methods are recommended in statistical theory but the maximum likelihood estimation method is the supreme used. Let X is random variable following Exponentiated QED distribution of size n with a vector of parameters$(\alpha, \lambda, q)^T$. Then sample likelihood function is given as:

$$\prod_{i=1}^{n} g(x_i) = \prod_{i=1}^{n} \alpha(2-q)\lambda e_q(-\lambda x_i)\left(1-[1+(q-1)\lambda x_i]^{\frac{2-q}{1-q}}\right)^{\alpha-1}$$

Log-likelihood function is

$$L = n\log\alpha + n\log(2-q)\lambda + \log e_q\left(-\lambda\sum_i x_i\right) + (a-1)\sum\log\left[1-[1+(q-1)\lambda x_i]^{\frac{2-q}{1-q}}\right]$$

The exact solution of the estimator is not possible. So it is well-situated to use Newton-Raphson algorithm to maximize the above likelihood function numerically. One can use R or MATHEMATICA.

5. APPLICATION TO TIME TO EVENT DATA

In this section, we provide a time to event (TTE) data analyses to assess the goodness-of-fit of QED and EQED distributions.The data set described by [16] represent the survival times of patients tribulation from Head and Neck cancer disease and treated by a combination of radiotherapy and chemotherapy for 44 patient.

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36
74.47	81.43	84	92	94	110	112	119	127	130
155	159	173	179	194	195	209	249	281	319
519	633	725	817	1776	36.47	133	339	68.46	140
432	78.26	146	469						

Table 3. TTE Survival Data

The maximum likelihood estimates (MLEs), the corresponding standard errors of the unknown parameter for the TTE data are presented Table 4.

	QED		EQED	
Estimate	Value	S.E	Value	S.E
$\hat{\lambda}$	0.0127	0.0042	0.0224	0.0124
$\hat{q}$	1.4162	0.0942	1.3595	0.0845
$\hat{\alpha}$	**	**	2.0293	0.6985

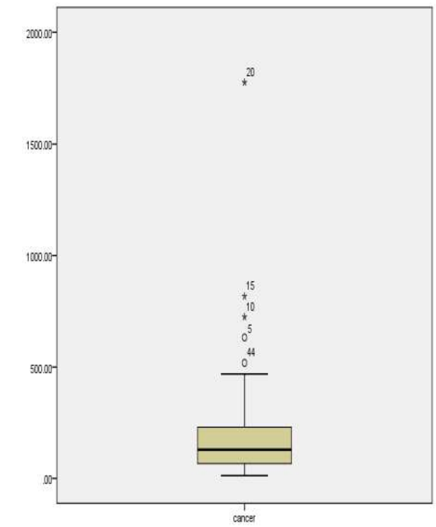

Figure3. Descriptive
Statistics of TTE data

	AIC	BIC	KS	A	P-value
QED	569.534	573.103	0.129	0.168	0.417
EQED	561.797	567.149	0.067	0.115	0.979

Table5. Goodness of fit tests

Several goodness of fit criterion were used to test if the data fit the model including, Akaike information criteria (AIC), Bayesian information criteria (BIC), and two distribution tests; K-S and Anderson-Darling (A-D).

The goodness of fit results was acceptable and all values for EQED is less than the goodness of fit tests of QED. The results indicate an excellent fit with K-S distance value between the empirical and the theoretical with P-values for QED and EQED equal to 0.48 and 0.98, respectively. The results indicated that adding a new parameter to the distribution leads to a better fit to the data.

CONCLUDING REMARKS

In this article, EQD is discussed and EQED is proposed. A mathematical treatment of the suggested distribution including some formulas for the probability density and distribution functions, hazard, reliability are provided. The formulas of the first fourth moments are given under some restrictions and the estimation of the parameters using by maximum likelihood method are given in the unclosed form. The usefulness of the suggested distribution is illustrated in an analysis of TTE data.

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INFORMATION-THEORETIC ESTIMATION APPROACH: TUTORIAL AND ILLUSTRATION

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ABSTRACT

In this tutorial, the information theoretic estimation approach as proposed by "Golan, A., G. Judge, D. Miller. (1996) [*Maximum entropy econometrics: Robust estimation with limited data.* New York: John Wiley and Sons]" for estimating a nonlinear regression model will be illustrated. The tutorial is divided into two parts; theoretical and empirical. The theoretical illustration will be used for estimating the unknown parameters of the quadratic regression model. However, the empirical illustration will study the performance of using different entropy measures (i.e., Shannon, Renyi and Tsallis) in estimating the probability of a discrete event.

Keywords: Generalized Maximum Entropy, Entropy Measures, Jayne's dice Problem, Nonlinear Regression.

1. INTRODUCTION

The problem of statistical inference is well known as a process of using data analysis to investigate the properties of an underling distribution. However, when the underling distribution is unknown we need advance statistical procedure for drawing inferences from limited and insufficient information. One of these statistical procedures was suggested by [15]; which consider the foundations of information theoretic approach in statistical inference or the inference under uncertainty. As a consequence,[11, 12] proposed a generalization of Bernoulli's and Laplace's principle of insufficient reason formulated based on the recognized work of [15].Jaynes's maximum entropy (ME) formalism aimed at solving any inferential problem witha well- defined hypothesis space and noiseless but incomplete information.This formalism was subsequently generalized to the linear model by [8]; who suggested the generalized maximum entropy (GME) estimation approach. Then after, many researchers extended and developed the idea of GME to several linear models [1, 2, 3, 4, 5, 6, 7, 8, 9]

In this paper, the information theoretic approaches ME and GME will be discussed in estimating the unknown distribution and in the context of the quadratic regression models, respectively.

The rest of this article is organized as follows, Section 2 the definition of the entropy will be given and some entropy measures will be defined. Section 3 an illustration of the GME estimation procedure in fitting the quadratic regression model. Section 4 will illustrate the Jayne's diceproblem in estimating the unknown distribution using different entropy measure. The article ends with a concluding remark section.

2. ENTROPY DEFINITION

Entropy as a mere word has a high diversity in meaning and also developed and used in many fields; the origin of it derived from the Greek meaning "transformation"; an important concept in thermodynamics/ physics which states that any change occurs spontaneously in a physical system must be accompanied by an increase in the amount of "entropy" here it means the amount of changing in a system[1].In earlier 1870's a statistical scientists gave "Entropy" a statistical meaning related to the probability theory such as Boltzman, Gibbs and Maxwell considering entropy as a measure of the information. In 1948, Shannon introduced the information theory (concept of it: having a way to transfer the data of any type or size without having any loss)how considered entropy as a fundamental concept and a basic measure in that precisely measures the amount of the data (in bit) including the error (which called uncertainty amount). The entropy can be measured by the maximum information that can be obtained from an event, at the same time; the information can bemeasured by the occurred probability of that event. Accordingly, many entropy measures can be define, for illustration as; let the X be a discrete random variable with K possible outcomes; say $x_1, x_2, \ldots., x_k$; where the probability of occurence of the jth outcome is $p_j; j = 1,2, \ldots, k$ such that $\sum_j p_j = 1$ (Figure.1)

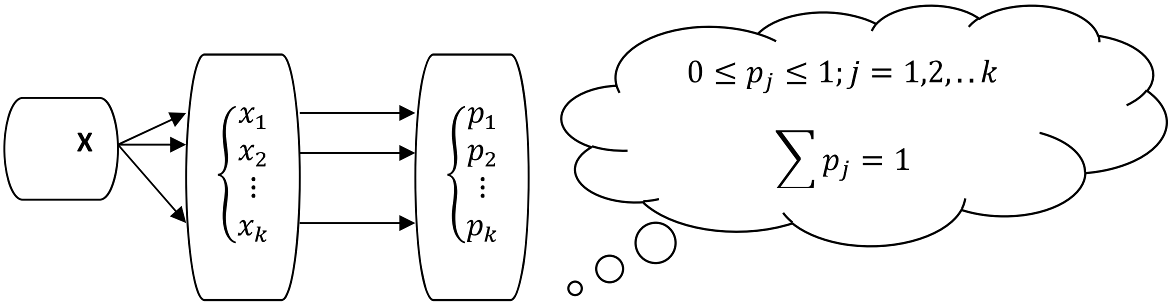

Figure 1. Illustration of a discrete random variable with finite probabilities

Then the information of a the j[th] event can be obtained as $\left\{h(x_j) = ln\left(\frac{1}{p_j}\right) = -ln(p_j)\right\}$; where the amount of information is defined as $\left\{h(x_j) = log_2\left(\frac{1}{p_j}\right) = -log_2(p_j)\right\}$. Accordingly, [15] defines the entropy as the expected information content of an outcomeof *X* with a discrete probability distribution *P*as H(P); Illustration is given in (Figure 2).

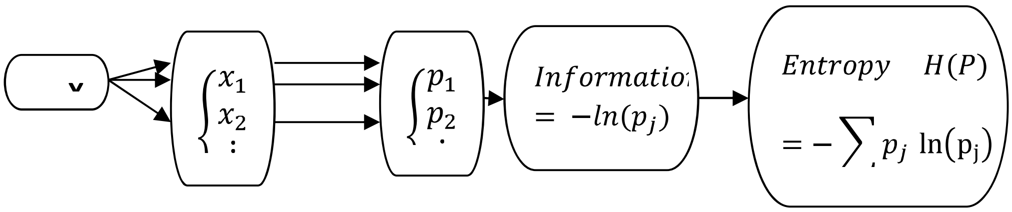

Figure 2. Illustration of Information Vs. Entropy

There are many poplar generalized entropy measures [14, 16], the most interesting and well-known in information theory are

- Renyi Entropy $\left\{R(\alpha) = \frac{1}{1-\alpha} \ln \sum_k p_k^{\alpha}\right\}$, where $\alpha > 0$; and
- Tsallis Entropy $\left\{T(q) = \frac{1}{1-q}\left[\sum_k p_k^{q} - 1\right]\right\}$, where $q > 0$

Noting that, both measures of order 1 are reduced to the Shannon Entropy, (Figure 3).

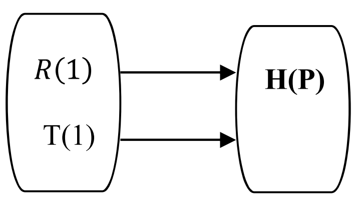

Figure 3. Relationships between Shannon Entropy and Renyi or Tsallis Entropies

3. FITTING QUADRATIC REGRESSION MODEL

Quadratic regression model is a polynomial regression model of order 2. In general, quadratic regression is a process of fitting parabola equation to a set of data which can be represented in the following equation [13]:

$$y_i = \alpha + \beta_1 x_i + \beta_2 {x_i}^2 + \epsilon_i; i = 1,2,\dots,n, \quad (1)$$

Whereα, β_1 and β_2 are the unknown parameters and y is the response variable while x is the explanatory variable. There are several estimation methods that can be used to fit "Eq. (1)".However; our interesting in this article is to use the GME as a new estimation method. Unlike the ME, the GME has an extra step in unknown parameters are not in probability forms before starting the estimation process. Therefore, Following [3, 9, 10] we should rewrite the unknown parameters given in Eq.(1) as a convex combination to a discrete random variable. Accordingly, the new formulation of the unknown parameters and the error term will be rewritten as:

$$\left\{ \alpha = \sum_{i=1}^{k} a_i p_i \quad , \quad \beta_1 = \sum_{j=1}^{r} b_{1j} q_{1j} \, , \ \beta_2 = \sum_{c=1}^{s} b_{2c} q_{2c} \text{ ,and } \varepsilon_t = \sum_{l=1}^{m} v_{tl} w_{tl} \right\}$$

It is worth to say here some values should be known to the researcher before he starts in the estimation, these values includes k, r, s and m which reflects the number of unknowns in the new parameterizations. Based on [3], the research can select these values between 3 and 7. Moreover, the realizations which include {a, b1, b2, and v} are given values thatdistributed uniformly around zero. Now, the new model will be of the form:

$$\left\{ y_t = \sum_{i=1}^{k} a_i p_i + \left(\sum_{j=1}^{r} b_{1j} q_{1j}\right) * x_t + \left(\sum_{c=1}^{s} b_{2c} q_{2c}\right) * x_t^2 + \sum_{l=1}^{m} v_{tl} w_{tl} \right\} \quad (2)$$

In this model we have *{k+r+s+m*n}* unknowns. However, based on the GME formulation we have *{3+m*n}* equations, therefore, "Eq. (2)"is an ill-posed models [7,8]. Using GME, the model can be estimated in four steps [1, 3]:

Step.1: Re-parametrize the unknown parameters and the disturbance term (if they are not in probabilities form) as a convex combination of expected value of a discrete random variable.
Step.2: Rewrite the model with the new re-parametrization.
Step.3: Formulate the GME problem as a nonlinear programming problem in the following form

Objective function = Entropy function

Subject to
(1) The re-parametrized model
(2) The Normalization constraints.

Step.4: Solve the nonlinear programming by using Lagrange method.

According to this algorithm the GME problem is

Maximize $H(p, q_{1j}, q_{2c}, w) = -\sum p_i \ln p_i - \sum q_{1j} \ln q_{1j} - \sum q_{2c} \ln q_{2c} - \sum w_{tl} \ln w_{tl}$

Subject to:

$$1 - y_t = \sum_{i=1}^{k} a_i p_i + \left(\sum_{j=1}^{r} b_{1j} q_{1j}\right) * x_t + \left(\sum_{c=1}^{s} b_{2c} q_{2c}\right) * x_t^2 + \sum_{l=1}^{m} v_{tl} w_{tl}$$

$$2 - \sum p_i = 1 \ ; \ \sum q_{1j} = 1 \ ; \ \sum q_{2c} = 1 \ ; \ \sum w_{tl} = 1$$

Now, we will use the Lagrangian method to solve this problem and find the appropriate estimates for each parameter as follows:

L = H(p, q_{1j}, q_{2c}, w)+$\lambda_1(y_t - \sum_{i=1}^{k} a_i p_i - \left(\sum_{j=1}^{r} b_{1j} q_{1j}\right) * x_t - (\sum_{c=1}^{s} b_{2c} q_{2c}) * x_t^2 - \sum_{l=1}^{m} v_{tl} w_{tl})$ +$\lambda_2(\sum p_i - 1)$ +λ_3 $(\sum q_{1j} - 1)$ + λ_4 $(\sum q_{2c} - 1)$ + λ_5 $(\sum w_{tl} - 1)$

Solving the first conditions, then we have:

$$p_i = \frac{e^{-\lambda_1 a_i}}{\sum_{i=1}^{k} e^{-\lambda_1 a_i}}, \quad q_{1j} = \frac{e^{-\lambda_1 x_t b_{1j}}}{\sum_{j=1}^{r} e^{-\lambda_1 x_t b_{1j}}}$$

$$q_{2c} = \frac{e^{-\lambda_1 x_t^2 b_{2c}}}{\sum_{c=1}^{s} e^{-\lambda_1 x_t^2 b_{2c}}}, w_{tl} = \frac{e^{-\lambda_1 v_{tl}}}{\sum_{l=1}^{m} e^{-\lambda_1 v_{tl}}}$$

This will be applied on a numerical optimization package as R or Matlab to have the desired results.

4. EMPIRICAL ILLUSTRATION: JAYNE'S DICE PROBLEM

In 1957and based on the information theory concept (Shannon, 1948), a new estimation method raised by Jayne's called the Maximum Entropy Principle (MEP) which estimated parameters based on finding a probability distribution subject to some constraints came up basically from the data .The estimator that revealed by this way is not necessarily the best one but it's the best depending on what information's we have. The estimation algorithm of ME is given by [1]. To illustrate this algorithm we revisited the Jayne's dice problem. The problem can be described as follows: When a dice is rollinga very large number of times "N", then the upper-face could be any value j such that j = 1,2, ..., 6 with corresponding probabilities $p_1, p_2, \dots, p_6$, such that $p_i \in [0, 1]$ and $\sum p_i = 1$. If we told that the average number of upper-faces was not 3.5 " which occurred with a fair dice", instead we assume the average to be "<a>" where a could be any real number between 1 and 6; that is to say $\{\sum_{i=1}^{6} i * p_i = a\}$. Then the problem is "what is the optimal distribution "probabilities of each event" in this experiment that satisfies both constraints. This is clearly an ill-posed problem which can be formulated based on the ME algorithm [1] as a nonlinear programming system(Figure 4).

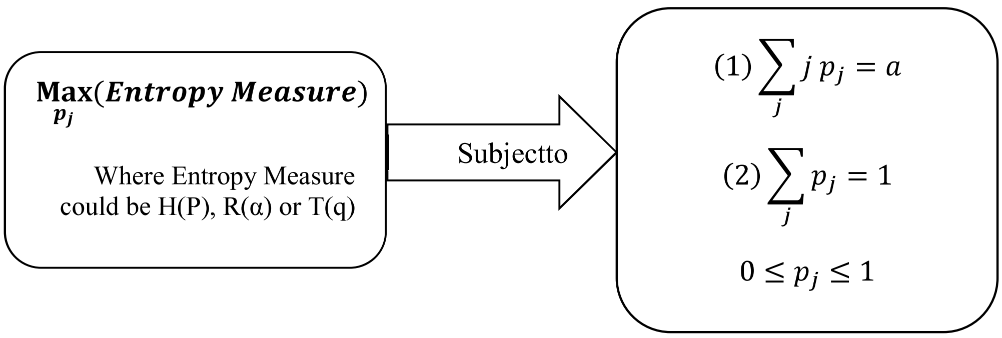

Figure 4. ME Mathematical programming system

The model given in Figure 4, can be solved by applying the lagrangian method. We solved this problem under the assumption that <a> = 2.5 or 4.0; the results are given in Table 1.

Table 1: optimal solution of Jayne's dice problem

Entropy Measure	<a>	p_1	p_2	p_3	p_4	p_5	p_6	$H(p)$
H(P)	2.5	0.346	0.239	0.165	0.114	0.079	0.055	1.61
	4.0	0.104	0.124	0.147	0.174	0.207	0.245	1.75
R(0.5)	2.5	0.368	0.225	0.152	0.109	0.082	0.064	1.70
	4.0	0.107	0.124	0.144	0.171	0.205	0.250	1.77
T(0.9)	2.5	0.352	0.237	0.162	0.113	0.079	0.057	1.77
	4.0	0.104	0.122	0.145	0.173	0.207	0.248	1.90

It could be noted that from Table 1, the entropy value of Shannon measure is less than other entropy measures. Also, the probability values decreases (that is to say $p_1 < p_2 < \ldots < p_6$) when the value of <a> less than 3.5; while the probability values increasing when <a> is more than 3.5.

CONCLUDING REMARKS

This article discussed the steps that should be used in fitting quadratic regression model by using the generalized maximum entropy estimation approach. The GME suggestsof reparametrize the regression model by rewriting the unknown parameters as expected values of a discrete random variable then go through four steps in order to estimate the unknown parameters. An illustration is given using the Jayne's dice problem, using different entropy measures, the results indicated that Shannon entropy is the best measure to be use for fitting equation to data in terms of minimizing the uncertainty of the estimator.

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CONSTRUCTING A NEW MIXED PROBABILITY DISTRIBUTION (QUASIY-LINDELY)

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ABSTRACT

In this paper the two parameters mixed probability distribution from exponential p.d.f. (β), and two parameters Gamma(β,α), which is called (Quasiy-Lindely) is introduced, the p.d.f. is defined and also the CDF and Risk function and hazard function are estimated using methods of moments and maximum likelihood and L-moment. The comparison is done through simulation using different values of sample size n and different set of initial values of parameters (β,α) and all the results obtain using (function fsolve) in program (MATLAB R2012a) and function x=fsolve(fun, xo) and all the results of estimation are explained in tables, and also conclusions and referenced are exposed.

Keywords: Two parameters Gamma (2, β); Moments method estimators (MOM); Maximum likelihood estimators (MLE).

1. INTRODUCTION

Quasi Lindely probability distribution is one of the mixed distributions for exponential with parameter (β) and Gamma distribution with two parameters (2,β) many researches work on introducing mixed distribution like Lindely [1] and Sankaran [3] introducing lindely with discrete Poisson also Gupta and Kundu[4] introduced generalized exponential with estimation as well as in Lindely[1] introduce fiducial distribution with applying bayes estimators to estimate Risk function and Lindely[2] compared different baysian estimator for parameters of lindely distribution shanker and Mishra [5] "introduced a paper about quasi lindely distribution, here we continue the work about this distribution and we apply three different methods like moments and L-moments and Maximum likelihood method to compare the Risk function of two parameters (quasi-lindely)[6][7][8].

2. THEORETICAL ASPECT

2.1 Quasi Lindely

It is one continuous distribution obtained from mixing:

$$f_1(x) = \beta\, e^{-\beta x} \qquad x>0 \tag{1}$$

Exponential distribution, and the second one is Gamma with (2, β):

$$f_2(x) = \beta^2\, x\, e^{-\beta x} \tag{2}$$

$$f(x,\alpha,\beta) = p f_1(x) + (1-p) f_2(x) \tag{3}$$

$$= \frac{\alpha}{\alpha+1}\, \beta e^{-\beta x} + \left(\frac{1}{\alpha+1}\right)\beta^2 x\, e^{-\beta x} \tag{4}$$

Equation (4) can be simplified to:

$$f(x,\alpha,\beta) = \frac{\beta(\alpha+\beta x)}{(\alpha+1)} e^{-\beta x} \qquad x>0,\ \beta>0,\ \alpha>-1 \tag{5}$$

β is scale parameter and α is location parameter. The p.d.f. equation (5) is called (quasi lindely) when (α=β) then p.d.f. (5) reduced to Gamma (2,β):

$$f(x,\beta) = \frac{\beta^2}{1+\beta}(1+x)e^{-\beta x} \qquad x>0,\ \beta>0$$

While the cumulative distribution function is:

$$F_x(x) = pr(X \le x) = \int_0^x f(t)\ dt = \frac{\beta}{\alpha+1}\int_0^x \left[\alpha\, e^{-\beta u} + \beta u e^{-\beta u}\right] du$$

Therefore the CDF of Quasi lindely is:

$$F_x(x) = 1 - \frac{(1+\alpha+\beta x)e^{-\beta x}}{\alpha+1} \qquad x>0,\ \beta>0,\ \alpha>-1 \qquad (6)$$

We can also prove that (mr), the rth formula about origin is:

$$mr = E(x^r) = \int_0^{\infty} x^r\, f\,(x,\alpha,\beta)dx \qquad (7)$$

Using transformation (x= y/β), we can solve integral (6) and prove that:

$$Mr = E(x^r) = \frac{\Gamma(r+1)[\,\alpha+r+1]}{(\alpha+1)\beta^r} \qquad (8)$$

From Mr we find:

$$E(x) = \acute{\mu}_1 = \frac{\alpha+2}{\beta(\alpha+1)} = \frac{\sum x_i}{n} \qquad (9)$$

And

$E(x^2) = \acute{\mu}_2 = \frac{2(\alpha+3)}{\beta^2(\alpha+1)} = \frac{\sum x_i^2}{n}$ Then the variance is:

$$\sigma^2 = E(x-\mu)^2 = \frac{\alpha^2+4\alpha+2}{\beta^2(\alpha+1)^2} \qquad (10)$$

And also we can find the coefficient of variation (C.V.)

$$\text{C.V.} = \frac{\sigma}{\acute{\mu}_1} = \sqrt{\frac{\alpha^2+4\,\alpha+2}{\alpha+2}} \qquad (11)$$

After we define the distribution and its mean and variance, we work on estimating its two parameters (β, α) by method of moments and then L-moments and maximum likelihood and then comparing estimators by simulation procedure and use these estimators $(\hat{\alpha},\hat{\beta})$ to estimate risk function h(t) which is:

h(t)= $\frac{f(t)}{s(t)}$ for human application and h(t)= $\frac{f(t)}{R(t)}$ for tools and equipments. In our studied probability distribution the hazard function:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\beta(\alpha+\beta t)}{1+\alpha+\beta t} \qquad t>0,\ \alpha>-1,\ \beta>0$$

and $MSE\left(\hat{h}(t)\right) = \frac{\sum_{i=1}^{n}(\hat{h}_i(t)-h(t))^2}{n}$

2.2 Moments estimator

The estimators by this method obtained from solving equation:

$\acute{\mu}_r = Ex^r$ for r=1, 2

$$\acute{\mu}_1 = \frac{\alpha+2}{\beta(\alpha+1)} = \frac{\sum_{i=1}^{n} xi}{n} = \frac{2(\hat{\alpha}+3)}{\beta^2(\alpha+1)} = \frac{\sum_{i=1}^{n} xi^2}{n}$$

Then $\sum x_i^{\,2}\,(\hat{\alpha}+1)\beta^2 = 2n\,(\hat{\alpha}+3)$

$$\widehat{\beta}_{mom} = \sqrt{\frac{2n\,(\hat{\alpha}+3)}{\sum x_i^{\,2}(\hat{\alpha}+1)}} \qquad (12)$$

This equation solved numerically by fixed point method.

According to given values of α *and* β and values $\{x_i\}$ at sample size (n), also we can use (function in f solve) in program (MATLAB r2012 a).

X= f solve (fun, x0)

Finding $\hat{\beta}_{mom}$, we can use it to find $\hat{\alpha}_{mom}$ from solving equation (13):

$$\overline{X} = \frac{\hat{\alpha}_{mom}+2}{\hat{\beta}_{mom}(\hat{\alpha}_{mom}+1)} \qquad (13)$$

$$\frac{\sum_{i=1}^{n} xi^2}{n} = \frac{2(\alpha+3)}{\beta^2(\alpha+1)}$$

$$\hat{\beta}_{mom} = \sqrt{\frac{2n\,\alpha+6n}{\sum x_i^{\,2}(\alpha+1)}} \qquad (14)$$

Solve by fixed point method to find the estimator $\hat{\beta}_{mom}$.

2.3 Estimation by L moments

This method is due to Hosking (1990) which depend on order statistics for expected value of liner components from order Statistics.

Here we have two parameter (β, α) so we need two linear moments were first find the formula of $\acute{\mu_r}$ from equation (14) (Linear moments).

$\acute{\mu_r} = \int x\,[F_{(X)}]^r f_{(X)}\,dx \ldots 14$

While linear moments for sample is:

$$L_1 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}$$

$$L_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} - L_1 (15)$$

And then we equate population moments μ_1, μ_2 with

$$E_{(X)} = L_1 (16)$$

$$E_{(X)}{}^2 = L_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} (17)$$

Now the estimator by L- Moments produce:

$$\frac{\frac{2(\alpha+3)}{(\alpha+2)^2}}{\bar{X}^2\,(\alpha+1)} = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} - L_1 (18)$$

$$\frac{2\bar{X}^2(\alpha+1)(\alpha+3)}{(\alpha+2)} = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} - L_1 (19)$$

Solve equation (19) numerically gives $(\hat{\alpha}_{L\,mom})$ then use $(\hat{\alpha}_{L\,mom})$introduce $(\hat{\beta}_{L\,mom})$

$$\frac{(\hat{\alpha}_{L\,mom}+2)}{\hat{\beta}_{L\,mom}\,(\hat{\alpha}_{L\,mom}+1)} = \bar{X} (20)$$

2.4 Maximum likelihood method

Let $x_1, x_2 \ldots x_n$ be ar.s from P.D.F in equation (5), Then:

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \beta) = \left(\frac{\beta}{\alpha+1}\right)^n \prod_{i=1}^{n} (\alpha + \beta x_i)\, e^{-\beta \sum_{i=1}^{n} x_i}$$

$$\log L = n\log(\beta) - n\log(\alpha+1) + \sum_{i=1}^{n}\log(\alpha+\beta x_i) - \beta\sum_{i=1}^{n} x_i \quad (21)$$

Then $\frac{\partial log}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{x_i}{(\alpha+\beta x_i)} - \sum_{i=1}^{n} x_i$ And $\frac{\partial log}{\partial \alpha} = \frac{-n}{\alpha+1} + \sum_{i=1}^{n}\frac{1}{(\alpha+\beta x_i)}$

From $\frac{\partial log}{\partial \alpha} = 0$, $\hat{\alpha}_{MLE} = \left(\frac{n}{\sum_{i=1}^{n}\frac{1}{(\alpha+\beta x_i)}}\right) - 1 = \left(\frac{n}{\sum_{i=1}^{n}(\alpha+\beta x_i)^{-1}} - 1\right) (22)$

$$\hat{\beta}_{MLE} = \left(\frac{n}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i(\hat{\alpha}+\hat{\beta} x_i)^{-1}}\right) (23)$$

3. SIMULATION PROCEDURES

We comparing three estimators of risk function by simulation procedure were the data is generated we assume sample size n=20, 40, 60, 80. And generate the values of (random variable x) which follows quasi lindely with two parameters (β, α) using method of (reject and accept) using the following steps:

1) Generate random variable Ui distributed uniformly ui~ u (0,1).
2) Generate another two random variables ziexp(β) and vi ~gamma (2, θ).
3) Let $p = \frac{\alpha}{\alpha+1}$ if ui$\leq p$ then $X_i=Z_i$ otherwise $X_i=V_i$
4) Estimate parameters of (Q.L) by (i) method of moments (ii) method of L-moments (iii) methods of maximum likelihood.
5) The comparison between estimators of $\hat{h}_i(t)$ is done using mean square error MSE, i.e.:$MSE\left(\hat{h}_i(t)\right) = \frac{\sum_{i=1}^{n}(\hat{h}_i(t)-h(t))^2}{n}$

We indicate that the sample size chosen are (n= 20, 40, 60, 80) and initial values of (β, α) are (α= 0.5, 1.2) and (β= 0.8, 1.5).

Table 1: Estimators of risk function

n	α	β	ti	$\hat{h}_{MOM}$	$\hat{h}_{LMOM}$	$\hat{h}_{MLE}$
			1.5	0.3378	0.3187	0.3062
			2.5	0.3978	0.3752	0.3698
20	0.5	0.8	3.5	0.4069	0.4612	0.4802
			4.5	0.3135	0.3051	0.3062
			5.5	0.3472	0.3321	0.3473
			1.5	0.3152	0.3224	0.3116
			2.5	0.4022	0.3252	0.3462
20	0.5	1.5	3.5	0.4632	0.3637	0.3725
			4.5	0.4031	0.4166	0.4235
			5.5	0.4421	0.4617	0.4382
			1.5	0.4152	0.4170	0.4231
			2.5	0.3613	0.4228	0.4116
20	1.2	0.8	3.5	0.3825	0.4107	0.4221
			4.5	0.3746	0.41106	0.4017
			5.5	0.4170	0.4005	0.4003
			1.5	0.4165	0.4325	0.4227
			2.5	0.4636	0.4265	0.4266
20	1.2	1.5	3.5	0.4601	0.4394	0.4278
			4.5	0.4421	0.4255	0.4166
			5.5	0.4392	0.4106	0.4005

Table 2: Estimators of Risk function $\hat{h}_i(ti)$ of Q.L.

n	α	β	ti	$\hat{h}_{MOM}$	$\hat{h}_{LMOM}$	$\hat{h}_{MLE}$
			1.5	0.4088	0.4188	0.3166
			2.5	0.4852	0.4663	0.3624
40	0.5	0.8	3.5	0.5321	0.4502	0.4088
			4.5	0.5506	0.4356	0.4521
			5.5	0.5563	0.5113	0.4312

Table 2 (Continued)

n	α	β	ti	$\hat{h}_{MOM}$	$\hat{h}_{LMOM}$	$\hat{h}_{MLE}$
			1.5	0.6141	0.5221	0.4025
			2.5	0.6233	0.5662	0.5763
40	0.5	1.5	3.5	0.6011	0.5582	0.5766
			4.5	0.5892	0.6043	0.5822
			5.5	0.5713	0.6122	0.5831
			1.5	0.5561	0.6003	0.3322
			2.5	0.5368	0.6112	0.3842
40	1.2	0.8	3.5	0.5311	0.6132	0.3226
			4.5	0.5677	0.6141	0.4205
			5.5	0.5078	0.6631	0.4762
			1.5	0.5146	0.4663	0.4612
			2.5	0.5526	0.4509	0.4663
40	1.2	1.5	3.5	0.5106	0.5403	0.5132
			4.5	0.5312	0.5266	0.5300
			5.5	0.5441	0.5466	0.5433

Table 3: Continue comparing estimators of hazard function of Q.L.

n	α	β	ti	$\hat{h}_{MOM}$	$\hat{h}_{LMOM}$	$\hat{h}_{MLE}$
			1.5	0.4335	0.4298	0.3274
			2.5	0.4902	0.4783	0.3752
60	0.5	0.8	3.5	0.5416	0.5206	0.3482
			4.5	0.5662	0.5703	0.4036
			5.5	0.5837	0.5663	0.4452
			1.5	0.3467	0.4076	0.3602
			2.5	0.4768	0.4767	0.3675
60	0.5	1.5	3.5	0.3202	0.5320	0.4217
			4.5	0.5388	0.3988	0.4452
			5.5	0.5702	0.6148	0.4906
			1.5	0.4224	0.4036	0.3263
			2.5	0.4736	0.4828	0.3862
60	1.2	0.8	3.5	0.3167	0.5166	0.4212
			4.5	0.5467	0.5467	0.4456
			5.5	0.6078	0.5782	0.4227
			1.5	0.3928	0.6122	0.3536
			2.5	0.4652	0.6037	0.6261
60	1.2	1.5	3.5	0.5088	0.5083	0.5142
			4.5	0.5436	0.6642	0.5521
			5.5	0.6036	0.6651	0.5136

Table 4: Comparing estimators of hazard function of Q.L.

n	α	β	ti	$\hat{h}_{MOM}$	$\hat{h}_{LMOM}$	$\hat{h}_{MLE}$
			1.5	0.3987	0.4036	0.3864
			2.5	0.4637	0.4726	0.4677
80	0.5	0.8	3.5	0.5082	0.5271	0.5022
			4.5	0.5392	0.5467	0.5536
			5.5	0.5514	0.5334	0.5542
			1.5	0.3886	0.4761	0.6019
			2.5	0.4617	0.5062	0.6211
80	0.5	1.5	3.5	0.5072	0.4582	0.5306
			4.5	0.5498	0.5563	0.5241
			5.5	0.5567	0.5571	0.5321
			1.5	0.6332	0.5572	0.5516
			2.5	0.6034	0.5862	0.5312
			3.5	0.6115	0.5599	0.5528
80	1.2	0.8	**Table 4 (Continued)**			
			4.5	0.6273	0.5603	0.5528
			5.5	0.6374	0.5432	0.5762
			1.5	0.3962	0.5531	0.5832
			2.5	0.4667	0.8054	0.6061
80	1.2	1.5	3.5	0.5132	0.6321	0.6364
			4.5	0.5416	0.6255	0.6472
			5.5	0.5521	0.6284	0.6566

Table 5: values of mean square error for estimating reliability function by three models

Model	N	MLE	MOM	BEST
	25	0.010976	0.010964	MOM
I	50	0.002115	0.005316	MLE
	75	0.00097	0.002987	MLE
	25	0.01664	0.01464	MOM
II	50	0.00403	0.00758	MLE
	75	0.00254	0.0055	MLE
	25	0.012014	0.009148	MOM
III	50	0.00342	0.00643	MLE
	75	0.001859	0.001992	MLE

CONCLUSIONS

(1). For three estimators of (α) and (β) by three different methods and then computing estimators of Risk function, we find that $\hat{R}_{MOM}$, $\hat{R}_{LMOM} = \frac{10}{80} * 100$ and $\hat{R}_{MLE} = \frac{43}{80} * 100$, and$\hat{R}_{MOM} = \frac{17}{80} * 100$

i.e. the first best one is MLE and then MOM and finally LMOM.

(2). In case of estimations in Reliability function we need to compute Reliability function for distribution of time to failure, but for biological application and medical applications we need to compare results by Risk function.

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NILPOTENT ELEMENTS AND EXTENDED SYMMETRIC RINGS [2]

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ABSTRACT

An endomorphism α of a ring R is called weak symmetric if whenever the product of any three elements of a ring R, abc, is a nilpotent element of R, then so is $ac\alpha(b)$. A ring R is called weak α-symmetric if there exist a weak symmetric endomorphism α of R. The notion of weak α-symmetric ring is a generalization of α-symmetric rings as well as an extension of symmetric rings. In this paper, we investigate characterization of weak α-symmetric and there related properties including extensions: In particular, we show that every semicommutative and weak α-symmetric ring is weak α-skew Armendariz. We also proved that, the semicommutative ring is weak α-symmetric if and only if the polynomial ring $R[x]$ of R is weak α-symmetric.

Keywords: semicommutative ring; symmetric ring; weak α-symmetric ring; weak α-skew Armendariz rings

1. INTRODUCTION

Throughout, R denotes as associative ring with unity. For a ring R with a ring endomorphism $\alpha: R \to R$, a skew polynomial ring $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. For a ring R, we denoted by $nil(R)$ the set of all nilpotent elements of R and by $R[x]$ the polynomial ring with an indeterminate x over R. A ring is called reduced if it has no nonzero nilpotent elements. Lambek called a ring R symmetric [8] provided $abc = 0$ implies $acb = 0$ for $a,b,c \in R$. Every reduced ring is symmetric ring [11, Lemma 1.1]. Cohn called a ring is reversible [3] if $ab = 0$ implies $ba = 0$ for $a,b \in R$, reversible rings are semicommutative, i.e., whenever $ab = 0$ we have $axb = 0$ for each element x of the ring, and semicommutative rings are abelian, namely, satisfy " idempotents are central " condition. Lambek called a right ideal I of a ring R symmetric if $rst \in I$ implies $rts \in I$ for all $r,s,t \in R$. If the zero ideal is symmetric then R is usually called symmetric. An endomorphis α of a ring R is called a weak reversible if whenever $ab \in nil(R)$ for $a,b \in R$, $b\alpha(a) \in nil(R)$. A ring R is called weak α-reversible if there exist a weak reversible endomorphism α of R [1]. A ring is said to be α-compatible if for each $a,b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$ [4]. According to Krempa [6], an endomorphism of a ring R is called to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring is called α-rigid if there exist a rigid endomorphism α of R. A ring R is α-rigid if and only if R is α-compatible [4, Lemma 2.2]. By [10], R is said to be weak α- rigid if $a\alpha(a) \in nil(R) \Leftrightarrow a \in nil(R)$. Also, a ring R is weak α-rigid and reduced if and only if R is α-rigid. An endomorphism of a ring R is called right (left) symmetric if whenever $abc = 0$ for $a,b,c \in R$, $ac\alpha(b) = 0$ $(\alpha(b)ac = 0)$. A ring is called right (left) α-symmetric if there exist a right (left) symmetric endomorphism α of R [7].

The notion of α-symmetric ring for an endomorphism α of a ring R is a generalization of α-rigid rings and an extension of symmetric rings. By [7, Theorem 2.8], a rings is α-rigid if and only if R is semiprime and right α-symmetric. Also, if the skew polynomial ring $R\,[x;\alpha]$ of a ring R is a symmetric ring then R is α-symmetric.

In this note, we introduce the concept of weak α-symmetric rings with respect to an endomorphism α of R. We considering the nilpotent elements instead of the zero element in α-symmetric rings to investigate the nilpotent elements in α-symmetric rings. We also investigate connections between weak α-symmetric condition and other related conditions such that α-

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symmetricity and weak α-rigidity. The relationship between α-compatible rings and weak α-symmetric rings is also studied. To illustrate the concepts and results some examples are included.

2. ON WEAK α-SYMMETRIC RINGS.

Definition 2.1. An endomorphism α of a ring R is called a weak symmetric if whenever abc ∈ nil(R) for a,b,c ∈ R, acα(b) ∈ nil(R). A ring is called weak α- symmetric if there exist a weak symmetric endomorphism α of R.

It is easy to see that any subring S with $\alpha(S) \subseteq S$ of a weak α-symmetricring is also weak α-symmetric. Also, if R is reduced ring then this definition coincides with the definition of α-symmetric ring [7].

The following example shows that there exists symmetric ring which is not weak α-symmetric for some endomorphism α of R.

Example 2.2. Let $R = S \oplus S$, where S be any non-zero symmetric ring. Then R is symmetric. Now, let $\alpha: R \to R$, given by $\alpha(a,b) = (b,a)$. For a =(1,0), b =(0,1), c =(1,1), $abc \in nil(R)$ but $ac\alpha(b) \notin nil(R)$. Therefore R is not weak α-symmetric.

For an endomorphism α of a ring R the map $\bar{\alpha}: T_n(R) \to T_n(R)$ defined by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$ for each $(a_{ij}) \in T_n(R)$ is a ring endomorphism of $T_n(R)$.

Proposition 2.3. *A ring R is weak α- symmetric if and only if the upper triangular matrix ring $T_n(R)$ over R is weak $\bar{\alpha}$-symmetric.*

Proof. One direction is trivial, since any subring S with $\alpha(S) \subseteq S$ of a weak α-symmetric is also weak α-symmetric. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij}) \in T_n(R)$ such that $ABC \in nil(T_n(R))$. Then $a_{ii}b_{ii}c_{ii} \in nil(R)$ for each $1 \leq i \leq n$. Since R is weak α-symmetric. Then $AC\bar{\alpha}(B) \in nil(T_n(R))$ and the result follows.

Recall that for a ring R and an (R,R)-bimodule N, the trivial extension of R by N is the ring $T(R,N) = R \oplus N$ with the usual addition and the multiplication $(r_1,n_1)(r_2,n_2) = (r_1r_2, r_1n_2 + r_2n_1)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & n \\ 0 & r \end{pmatrix}$, where $r \in R$ and $n \in N$ and the usual matrix operations are used.

Corollary 2.4. *Let α be an endomorphism of a ring R. Then R is weak α-symmetric if and only if T(R,R) is weak $\bar{\alpha}$-symmetric.*

It is clear that any weak α-symmetric ring is weak α-reversible. Since every n-by-n full matrix ring $M_n(R)$ over a weak α-reversible is not weak $\bar{\alpha}$-reversible [1,Example 2.5]. Then every n-by-n full matrix ring $M_n(R)$ over weak α-symmetric is not weak $\bar{\alpha}$-symmetric, where $n \geq 2$.

Proposition 2.5. *Let R be a ring with an endomorphism α.*

(1) If α is a monomorphism, then each weak α-symmetric ring is weak α-rigid.

(2) If $nil(R)$ is a symmetric ideal, then each weak α-rigid is weak α-symmetric.

Proof.

(1) Let $a\alpha(a) \in nil(R)$. Then $\alpha(a)\alpha(a) = \alpha(a^2) \in nil(R)$, since R is weak α-symmertic. There exist $k > 0$ such that $\alpha(a^{2k}) = 0$. Hence $a \in nil(R)$, since α is a

monomorphism. Conversely, let $a \in nil(R)$ then $a\alpha(a) \in nil(R)$, because R is weak α-symmetric.

(2) Let $abc \in nil(R)$, then $cab \in nil(R)$ and $\alpha(c)\alpha(a)\alpha(b) \in nil(R)$, hence $\alpha^2(b)\alpha(c)\alpha(a)\alpha(b)ca \in nil(R)$ since $nil(R)$ is an ideal. So $\alpha(b)ca \in nil(R)$ because R is weak α-rigid. Hence $a\alpha(b)c \in nil(R)$ and $ac\alpha(b) \in nil(R)$, since $nil(R)$ is a symmetric ideal.

Lemma 2.6. *Let R be a weak α-symmetric ring if $abc \in nil(R)$, then $a\alpha^n(b)c \in nil(R)$ and $\alpha^m(a)bc \in nil(R)$, for any positive even integers n,m.*

Proof. Let $abc \in nil(R)$. Since R is weak α-symmetric ring, then $ac\alpha(b) \in nil(R)$ and $c\alpha(b)a \in nil(R)$. By using again the weak α-symmetricity, we have $ca\alpha^2(b) \in nil(R)$ and $a\alpha^2(b)c \in nil(R)$, then $ac\alpha^3(b) \in nil(R)$ and $ca\alpha^4(b) \in nli(R)$, hence $a\alpha^4(b)c \in nil(R)$. Continuing this process we get $a\alpha^n(b)c \in nil(R)$ where n is an even positive integer. On the other hand, if $abc \in nil(R)$ then $cab \in nil(R)$, using the above method for cab, we get $bc\alpha^m(a) \in nil(R)$, hence $\alpha^m(a)bc \in nil(R)$ where m is a positive even integer.

Proposition 2.7. *For any weak α-symmetric ring R, we have the following statements:*

(1) If α is a monomorphism, then $\alpha(1) = 1$.

(2) $\alpha(1) = 1$ if only and only if $\alpha(e) = e$, for any central idempotent $e \in R$.

Proof .

(1) Suppose that α is a monomorphism of a ring$R$. Then $1(1-\alpha(1))\alpha(1) = 0, \alpha(1)\alpha(1-\alpha(1)) \in nil(R)$, since R is weak α-symmetric, then $\alpha(1-\alpha(1)) \in nil(R)$. Since α is a monomorphism, then $1-\alpha(1) \in nil(R)$. Note that $1-\alpha(1)$ is an idempotent of R, and then we get $1-\alpha(1) = 0$. So $\alpha(1) = 1$.

(2) Let e be a centeral idempotent in R, then $1(1-e)e = 0 \in nil(R)$. Hence $1(e)\alpha(1-e) \in nil(R)$. Thus there exists $n > 0$ such that $0 = (e\,\alpha(1-e))^n = e\alpha(1-e)$. Then $e(1-\alpha(e)) = e - e\alpha(e) = 0$, so $\alpha(e) = e\alpha(e)$. Similarly $1e(1-e) = 0 \in nil(R)$ and this implises $(1-e)\alpha(e) = 0$. Thus, $\alpha(e) = e\alpha(e)$. Therefore $\alpha(e) = e$. The converse is clear.

Theorem 2.8. *Let R be an abelian ring with $\alpha(e) = e$ for any $e^2 = e \in R$. Then the following statements are equivalent:*

(1) R is a weak α-symmetric ring.

(2) eR and $(1-e)R$ are weak α-symmetric.

Proof. Since any subring S with $\alpha(S) \subseteq S$ of a weak α-symmetric ring is also weak α-symmetric, so we will prove (2) ⇒ (1). Let a,b,$c \in R$ such that $abc \in nil(R)$. Then $eaebec \in nil(R)$ and $(1-e)a(1-e)b(1-e)c \in nil(R)$. Since eR and $(1-e)R$ are weak α-symmetric, then $eaec\alpha(eb) \in nil(R)$ and $(1-e)a(1-e)c\alpha((1-e)b) \in nil(R)$. Hence $eaec\alpha(eb) + (1-e)a(1-e)c\alpha((1-e)b) = eac\alpha(b) + (1-e)ac\alpha(b) = ac\alpha(b) \in nil(R)$. Therefore R is weak α-symmetric ring.

Let α be an endomorphism of a ring R. An ideal I of a ring R is said to be α-stable if $\alpha(I) \subseteq I$. If I is an α-stable ideal then $\bar{\alpha}: R/I \to R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring R/I [1].

Proposition 2.9. Let I be an α-stable and weak α-symmetric ideal of R. If $I \subseteq nil(R)$, then R/I is a weak $\bar{\alpha}$-symmetric ring if and only if R is a weak α-symmetric.

Proof. Assume that R/I is weak $\bar{\alpha}$-symmetric. Let $abc \in nil(R)$ for a,b,$c \in R$. then $\bar{a}\bar{b}\bar{c} \in nil(R/I)$. Thus $\bar{a}\bar{c}\alpha(\bar{b}) \in nil(R/I)$, since R/I is weak $\bar{\alpha}$-symmetric. So there exists a positive integer n such that $\left(ac\alpha(b)\right)^n \in I$, then $\left(ac\alpha(b)\right)^n \in nil(R)$. Therefore R is weak α-symmetric.

Conversely, suppose $\bar{a}\bar{b}\bar{c} \in nil(R/I)$. Then there exists a positive integer m such that $(abc)^m \in I$. Since $I \subseteq nil(R), abc \in nil(R)$. Thus $ac\alpha(b) \in nil(R)$ since R weak α-symmetric. Hence $\bar{a}\bar{c}\alpha(\bar{b}) \in nil(R/I)$ and R/I is weak $\bar{\alpha}$-symmetric.

By [11], a ring R is called an Armendariz ring if whenever $f(x)g(x) = 0$ where $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$, then $a_ib_j = 0$ for each i, j. Liu and Zhao [9] introduced weak-Armendariz rings. A ring R is called weak-Armendariz ring if whenever polynomials $g(x) = a_0 + a_1x + \cdots + a_nx^n$, $h(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $g(x)h(x) = 0$, then $a_ib_j \in nil(R)$ for each i,j. Each semicommutative ring is weak-Armendariz by [9].

The Armendariz property of ring was extended to one of skew polynomials [5]. A ring R is called α-skew Armendariz if for $g(x) = b_0 + b_1x + \cdots + b_nx^n$, $h(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x;\alpha]$ satisfy $g(x)h(x) = 0$ then $b_i\alpha^i(a_j) = 0$ for all $0 \le i \le n$ and $0 \le j \le m$ [5 ,Definition]. Zhang and chen introduce and studied weak α-skew Armendariz rings. A ring R is called weak α-skew Armendariz ring if for $g(x) = a_0 + a_1x + \cdots + a_nx^n, f(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $g(x)f(x) = 0$, then $a_i\alpha^i(b_j) \in nil(R)$ for all $0 \le i \le n$ and $0 \le j \le m$ [13].

Theorem 2.10. *Let R be a semicommutative ring. Then R is weak α-symmetric if and only if so is $R[x]$.*

Proof . Since any subring S with $\alpha(S) \subseteq S$ of weak α-symmetric is also weak α-symmetric, so we only prove $R[x]$ is weak α-symmetric when R is weak α-symmetric. Let $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ and $h(x) = \sum_{l=0}^{k} c_lx^l$ such that$f(x)g(x)h(x) \in nil\ R[x]$. Since R is semicommutative, then by [1,corollary 2.17], we have the following equations:

$$a_0b_0c_0 \in nil(R) \quad (1)$$

$$a_0b_1c_0 + a_0b_0c_1 + a_1b_0c_0 \in nil(R) \quad (2)$$

$$\vdots$$

$$a_0\, b_{v-2}c_1 + a_1b_{v-3}c_1 + a_2b_{v-4}c_1 + \cdots + a_{v-1}b_0c_0 \in nil(R) \quad (3)$$

$$\sum_{i+j+l=v} a_ib_jc_l \in nil(R) \quad (4)$$

Since R is semicommutative, $nil(R)$ is an ideal of R by [9, Lemma 3.1]. since $a_0b_0c_0 \in nil(R)$ then$c_0a_0b_0 \in nil(R)$, if we multiply the Eq. (1) from the left by c_0, then it follows:

$$c_0a_0b_1c_0 + c_0a_0b_0c_1 + c_0a_1b_0c_0 \in nil(R)$$

So,

$$c_0a_0b_1c_0 + c_0a_1b_0c_0 \in nil(R) \quad (5)$$

Now if we multiply the Eq. (5) by a_0 from right side, we can get $c_0a_0b_1c_0a_0 + c_0a_1b_0c_0a_0 \in nil(R)$, so $c_0a_0b_1c_0a_0 \in nil(R)$ and $c_0a_0b_1 \in nil(R) \Rightarrow a_0b_1c_0 \in nil(R)$, hence,

$$a_0b_0c_1 + a_1b_0c_0 \in nil(R) \quad (6)$$

By multiply the Eq. (6) by c_0 from left side, then it follows, $c_0a_0b_0c_1 + c_0a_1b_0c_0 \in nil(R)$ and $c_0a_1b_0c_0 \in nil(R)$, so $a_1b_0c_0a_1b_0c_0 \in nil(R)$ and $a_1b_0c_0 \in nil(R)$, then $a_0b_0c_1 \in nil(R)$. Now suppose that v is a positive integer such that $a_ib_jc_k \in nil(R)$ when $i + j + k < v$, we will show that $a_ib_jc_k \in nil(R)$ when $i + j + k = v$. If we multiply the Eq. (4) from the left side by c_0, then it follows that $\sum_{i+j+k=v} c_0 a_i b_j c_k \in nil(R)$. By induction hypothesis, $c_0 a_i b_j \in nil(R)$ whenever $i+j < v$. So $\sum_{i+j=v} c_0 a_i b_j c_0 \in nil(R)$, again multiply $\sum_{i+j=v} c_0 a_i b_j c_0$ by b_0 from the left side, we get $b_0 c_0 a_i \in nil(R)$ when $i < v$ and $b_0 c_0 a_i b_j c_0 \in nil(R)$ if $i < v$, hence $b_0 c_0 a_v b_0 c_0 \in nil(R)$ and $a_v b_0 c_0 \in nil(R)$. Now, $b_j c_k a_i \in nil(R)$ and $c_k a_i b_j \in nil(R)$ when $j+k+i < v$. So we can use same argument as above to get $a_0 b_v c_0 \in nil(R)$ and $a_0 b_0 c_v \in nil(R)$, we conclude that

$$\sum_{i+j+k=v} a_i b_j c_k \in nil(R) \qquad (7)$$

For all $0 \le i < v,\ 0 \le j < v$ and $0 \le k < v$.

Using again the induction hypothesis, $a_i b_j c_k \in nil(R)$ for $0 \le i < v,\ 0 \le j < v\ , 0 \le k < v$ and $j+k+i = v$. Hence $a_i b_j c_k \in nil(R)$ for each i,j,k. So $a_i c_k \alpha(b_j) \in nil(R)$ since R is weak α-symmetric. Thus $fh\bar{\alpha}(g) \in nil\ R[x]$ by [1,corollary 2.17]. Therefore $R[x]$ is weak α-symmetric.

Theorem 2.11. *Let α be an endomorphism of a ring R. If R is semicommutative and α-compatible ring. Then the ring $R[x;\alpha]$is weak α-symmetric.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ and $h(x) = \sum_{l=0}^{k} c_l x^l$ such that $f(x)g(x)h(x) \in nil\ R[x;\alpha]$. Since R is semicommutative, then by [1, proposition 2.16] we have the following equations:

$$a_0 b_0 c_0 \in nil(R)$$

$$a_0 b_0 c_1 + a_0 b_1 \alpha(c_0) + a_1 \alpha(b_0) c_0 \in nil(R) \qquad (8)$$

$$\vdots$$

$$a_{m-1}\alpha^{m-1}(b_n)\alpha^n(c_k) + a_m \alpha^m(b_{n-1})\alpha^{n-1}(c_k) + a_m \alpha^m(b_n)\alpha^n(c_{k-1}) \in nil(R) \qquad (9)$$

$$a_m \alpha^m(b_n)\alpha^n(c_k) \in nil(R) \qquad (10)$$

Since R is semicommutative, $nil(R)$ is an ideal of R by [9, Lemma 3.1]. since $a_0 b_0 c_0 \in nil(R)$, then $b_0 c_0 a_0 \in nil(R)$. R is weak α-reversible, hence $c_0 a_0 \alpha(b_0) \in nil(R)$, then $\alpha(b_0) c_0 a_0 \in nil(R)$. if we multiply the Eq. (8) from the right side by a_0, then it follows that : $a_0 b_0 c_1 a_0 + a_0 b_1 \alpha(c_0) a_0 + a_1 \alpha(b_0) c_0 a_0 \in nil(R)$. Then $a_0 b_0 c_1 a_0 + a_0 b_1 \alpha(c_0) a_0 \in nil(R)$. If we multiply the equation above by b_0 from right side, we have $a_0 b_0 c_1 a_0 b_0 + a_0 b_1 \alpha(c_0) a_0 b_0 \in nil(R)$ and since $a_0 b_0 c_0 \in nil(R)$ it follows $a_0 b_0 \alpha(c_0) \in nil(R)$. so $\alpha(c_0) a_0 b_0 \in nil(R)$ and we get $a_0 b_0 c_1 a_0 b_0 \in nil(R)$, then $a_0 b_0 c_1 \in nil(R)$, so $a_0 b_1 \alpha(c_0) + a_1 \alpha(b_0) c_0 \in nil(R)$ again by multiplying this equation by a_0 from the right side, we get $a_0 b_1 \alpha(c_0) a_0 \in nil(R)$ so $\alpha(c_0) a_0 b_1 \in nil(R)$ and $a_0 b_1 \alpha(c_0) \in nil(R)$, hence $a_1 \alpha(b_0) c_0 \in nil(R)$. continuing this process we have $a_i \alpha^i(b_j) \alpha^j(c_k) \in nil(R)$ for each i,j. Since R is α-compatible, $a_i \alpha^i(b_j) c_k \in nil(R)$ and $c_k a_i \alpha^i(b_j) \in nil(R)$, hence $c_k a_i b_j \in nil(R)$ and $b_j c_k a_i \in nil(R)$. Since R is semicommutative, $b_j a_i c_k a_i \in nil(R)$ and $b_j a_i c_k b_j a_i c_k \in nil(R)$ so $b_j a_i c_k \in nil(R)$ and $c_k b_j a_i \in nil(R)$, hence $b_j a_i \alpha^i(c_k) \in nil(R)$ and $a_i \alpha^i(c_k) \alpha^t(b_j) \in nil(R)$ for each i,j,k and t by weak α-reversibleity of R. Therefore $f(x)h(x)\bar{\alpha}g(x) \in nil\ R[x;\alpha]$ and the result follows.

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ON GENERALIZED $p-$VALENT NON-BAZILEVIˇC FUNCTIONS OF ORDER $\alpha+i\beta$

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ABSTRACT

In this paper, we introduce a subclass $N^{n}_{p,\mu}(\alpha,\beta,A,B)$ of $p-$valentnon-Bazileviˇc functions of order $\alpha+i\beta$. Some subordination relations and the inequality properties of $p-$valent functions are discussed. The results presented here generalize and improve some known results.

Keywords: Analytic functions;non-Bazileviˇcfunctions; differential subordination.

1. INTRODUCTION AND PRELIMINARIES

Let A_p denote the class of functions of the form

$$f(z)=z^{p}+\sum_{k=n}^{\infty}a_{k+p}z^{k+p}\ \left(p,n\in\mathbb{N}=\{1,2,3,...\}\right), (1.1)$$

which are analytic and $p-$valent in the open disc $U=\{z\in C:\ |z|<1\}$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, and we write:

$$f\prec g \text{ in } U \text{ or } f(z)\prec g(z),\ z\in U, \qquad (1.2)$$

if there exists a Schwarz function $w(z)$, which is analytic in U with $|w(0)|=0$ and $|w(z)|<1,\ z\in U$, such that $f(z)=g(w(z))$, $z\in U$.

Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, see Miller &Mocanu ([3], [4]), $f(z)\prec g(z)\Leftrightarrow f(0)=g(0)$ and $f(U)\subset g(U)$.

We define a subclass of A_p as follows:

Definition 1.1. Let $N^{n}_{p,\mu}(\alpha,\beta,A,B)$ denote the class of functions $f(z)\in A_p$ satisfying the inequality:

$$\left\{(1+\mu)\left(\frac{z^{p}}{f(z)}\right)^{\alpha+i\beta}-\mu\left(\frac{zf^{/}(z)}{pf(z)}\right)\left(\frac{z^{p}}{f(z)}\right)^{\alpha+i\beta}\right\}\prec\frac{1+Az}{1+Bz},\ z\in U\ (1.3)$$

where $\mu\in\mathbb{C},\ \alpha\geq\ \ \beta\in\mathbb{R},\ -\ \ \leq B\leq\ \ A\in\mathbb{R},\ A\neq B,$ and $p\in\mathbb{N}$. All the powers in (1.3) are principal values.

We say that the function $f(z)$ in this class is $p-$valentnon-Bazileviˇc functions of type $\alpha+i\beta$.

Definition 1.2. Let $f\in N^{n}_{p,\mu}(\alpha,\beta,\rho)$ if and only if $f(z)\in A_p$ and it satisfies:

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$$\operatorname{Re}\left\{(1+\mu)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}-\mu\left(\frac{zf^{/}(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right\}>\rho,\ (z\in U)\quad(1.4)$$

where $\mu\in\mathbb{C},\ \alpha\geq 0,\ \beta\in\mathbb{R},\ 0\leq\rho<p$ and $p\in\mathbb{N}$.

Special Cases:

(1) When $p=1$, then $N^n_{1,\mu}(\alpha,\beta,A,B)$ is the class studied by AlAmoush and Darus [6].

(2) When $p=1,\ \beta=0$, then $N^n_{1,\mu}(\alpha,0,A,B)$ is the class studied by Wang et al [1].

(3) When $p=1,\ \beta=0,\ \mu=-1,\ A=1$ and $B=-1$, then $N^n_{\mu}(\alpha)$ is the class studied by Obradovic [10].

(4) When $p=1,\ \beta=0,\ \mu=B=-1$ and $A=1-2\rho$ then $N^n_{1,-1}(\alpha,0,1-2\rho,-1)$ reduces to the class of non-Bazileviˇc functions of order $\rho(0\leq\rho<1)$. The Fekete-Szegö problem of the class $N^n_{1,-1}(\alpha,0,1-2\rho,-1)$ were considered by Tuneski andDarus [2].

We will need the following lemmas in the next section.

Lemma 1.3. [7] Let the function $h(z)$ be analytic and convex in U with $h(0)=1$. Suppose also that the function $\Phi(z)$ given by $\Phi(z)=1+c_nz^n+c_{n+1}z^{n+1}+\cdots$
is analytic in U.

If $\Phi(z)+\frac{1}{\gamma}z\Phi^{/}(z)\prec h(z)\ (z\in U\quad \gamma\geq\quad \gamma\neq\ )$ (1.5)

then

$\Phi(z)\prec\Psi(z)=\frac{\gamma}{n}z^{\frac{-\gamma}{n}}\int_0^z t^{\frac{\gamma}{n}-1}h(t)dt\prec h(z)$, and $\Psi(z)$ is the best dominant for the differential subordination (1.5).

Lemma 1.4. [8]Let $-1\leq B_1\leq B_2<A_2<A_1\leq 1$, then $\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}$.

Lemma 1.5. [9] Let $\Phi(z)$ be analytic and convex in U, $f(z)\in A_p$. If $f(z)\prec\Phi(z)$, $g(z)\prec\Phi(z),\ 0\leq\mu\leq 1$ then $\mu f(z)+(1-\mu)g(x)\prec\Phi(z)$.

Lemma 1.6. [11]Let $q(z)$ be a convex univalent function in U and let $\sigma\in\mathbb{C},\ \eta\in\mathbb{C}-\{0\}$ with

$$\operatorname{Re}\left\{1+\frac{zq^{//}(z)}{q^{/}(z)}\right\}>\max\left\{0,-\operatorname{Re}\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $\Phi\ z$ is analytic in U and $\sigma\Phi(z)+\eta z\Phi^{/}(z)\prec\sigma q(z)+\eta zq^{/}(z)$, then, $\Phi(z)\prec q(z)$ and $q(z)$ is the best dominant.

We employ techniques similar to these used earlier by Yousef et al. [13], Amourah et al. ([14], [15]), AlAmoush and Darus [16] and Al-Hawary et al. [13].

In the present paper, we shall obtain results concerning the subordination relations and inequality properties of the class $N^n_{p,\mu}(\alpha,\beta,A,B)$. The results obtained generalize therelated works of some authors.

2. MAIN RESULT

Theorem 2.1. Let $\mu \in \mathbb{C},\ \alpha \ge 0,\ \beta \in \mathbb{R},\ \alpha + i\beta \ne 0,\ -1 \le B \le 1,\ A \in \mathbb{R},\ A \ne B,$ and $p \in \mathbb{N}.$ If $f(z) \in N_{p,\mu}^{n}(\alpha, \beta, A, B)$, Then

$$\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec \frac{p(\alpha+i\beta)}{\mu n}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{p(\alpha+i\beta)}{\mu n}-1} du \prec \frac{1+Az}{1+Bz}. \ (2.1)$$

Proof. Let

$$\Phi(z) = \left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}. \ (2.2)$$

Then $\Phi(z)$ is analytic in U with $\Phi(0) = 1$. Taking logarithmic differentiation of (2.2) in both sides, we obtain $p(\alpha+i\beta) - (\alpha+i\beta)\frac{zf'(z)}{f(z)} = \frac{z\Phi'(z)}{\Phi(z)}$.

In the above equation, we have $1 - \frac{zf'(z)}{pf(z)} = \frac{1}{p(\alpha+i\beta)}\frac{z\Phi'(z)}{\Phi(z)}$.

From this we can easily deduce that

$$\left\{(1+\mu)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right\}. \ (2.3)$$

On a class of $p-$valentnon-Bazileviˇc functions

$$\Phi(z) + \frac{\mu z\Phi'(z)}{p(\alpha+i\beta)} \prec \frac{1+Az}{1+Bz}. (2.4)$$

Now, by Lemma 1.3 for $\gamma = \frac{p(\alpha+i\beta)}{\mu}$, we deduce that

$$\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec q(z) = \frac{p(\alpha+i\beta)}{\mu n} z^{-\frac{p(\alpha+i\beta)}{\mu n}} \int_0^z t^{\frac{p(\alpha+i\beta)}{\mu n}-1}\left(\frac{1+At}{1+Bt}\right)dt.$$

Putting $t = zu \Rightarrow dt = zdu$. Then we have the above equation with

$\frac{p(\alpha+i\beta)}{\mu n}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{p(\alpha+i\beta)}{\mu n}-1} du \prec \frac{1+Az}{1+Bz}$, and the proof is complete.

Corollary 2.2. Let $\mu \in \mathbb{C},\ \alpha \ge 0,\ \beta \in \mathbb{R},\ \alpha + i\beta \ne 0,\ \rho \ne$ and $p \in \mathbb{N}$. If $f(z) \in A_p$ satisfies

$$(1+\mu)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+(1-2\rho)z}{1-z},$$

then

$$\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec \frac{p(\alpha+i\beta)}{\mu n} z^{-\frac{p(\alpha+i\beta)}{\mu n}} \int_0^1 \frac{1+(1-2\rho)zu}{1-zu} u^{\frac{p(\alpha+i\beta)}{\mu n}-1} du,$$

or equivalent to

$$\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec \rho + \frac{p(\alpha+i\beta)(1-\rho)}{\mu n}\int_0^1 \frac{1+zu}{1-zu} u^{\frac{p(\alpha+i\beta)}{\mu n}-1} du.$$

Corollary 2.3. Let $\mu \in \mathbb{C},\ \alpha \geq 0,\ \beta \in \mathbb{R},\ \alpha + i\beta \neq 0,\ \operatorname{Re}\{\mu\} \geq 0$ and $p \in \mathbb{N}$, then $N_{p,\mu}^{n}(\alpha,\beta,A,B) \subset N_{p,0}^{n}(\alpha,\beta,A,B)$.

Theorem 2.4. Let $0 \leq \mu_1 \leq \mu_2,\ \alpha \geq 0,\ \beta \in \mathbb{R},\ \alpha + i\beta \neq 0,\ -1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, and $p \in \mathbb{N}$, then $N_{p,\mu_2}^{n}(\alpha,\beta,A_2,B_2) \subset N_{p,\mu_1}^{n}(\alpha,\beta,A_1,B_1)$. (2.5)

Proof. Suppose that $f(z) \in N_{p,\mu_2}^{n}(\alpha,\beta,A_2,B_2)$ we have $f(z) \in A_p$ and

$$\left\{(1+\mu_2)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu_2\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right\} \prec \frac{1+A_2 z}{1+B_2 z},$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore it follows from Lemma 1.4 that

$$\left\{(1+\mu_2)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu_2\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right\} \prec \frac{1+A_1 z}{1+B_1 z}, \quad (2.6)$$

that is $f(z) \in N_{p,\mu_2}^{n}(\alpha,\beta,A_1,B_1)$. So Theorem 2.4 is proved when $\mu_1 = \mu_2 \geq 0$.

When $\mu_1 > \mu_2 > 0$, then we can see from Corollary 2.3 that $f(z) \in N_{p,0}^{n}(\alpha,\beta,A_1,B_1)$, then

$$\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+A_1 z}{1+B_1 z}. \quad (2.7)$$

But

$$\left\{(1+\mu_1)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu_1\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right\}$$

$$=\left\{\left(1-\frac{\mu_1}{\mu_2}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} + \frac{\mu_1}{\mu_2}\left((1+\mu_1)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu_1\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right)\right\}.$$

It is obvious that $\frac{1+A_1 z}{1+B_1 z}$ is analytic and convex in U. Sowe obtain fromLemma 1.5 anddifferential subordinations (2.6) and (2.7) that

$$\left\{(1+\mu_1)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta} - \mu_1\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha+i\beta}\right\} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is, $f(z) \in N_{p,\mu_1}^{n}(\alpha,\beta,A_1,B_1)$. Thuswe have $N_{p,\mu_2}^{n}(\alpha,\beta,A_2,B_2) \subset N_{p,\mu_1}^{n}(\alpha,\beta,A_1,B_1)$.

Corollary 2.5. LetLet $0 \leq \mu_1 \leq \mu_2,\ \alpha \geq 0,\ \beta \in \mathbb{R},\ \alpha + i\beta \neq 0,\ 0 \leq \rho_1 \leq \rho_2$, and $p \in \mathbb{N}$, then $N_{p,\mu_2}^{n}(\alpha,\beta,\rho_2) \subset N_{p,\mu_1}^{n}(\alpha,\beta,\rho_1)$.

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SOLVING LOGISTIC EQUATION OF FRACTIONAL ORDER USING THE REPRODUCING KERNEL HILBERT SPACE METHOD[3]

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ABSTRACT

In this paper, we apply an efficient algorithm based on the reproducing kernel Hilbert space method (RKHSM) to solve a fractional version of the non-linear logistic differential equation. The fractional derivative is presented in the Caputo sense. In order to show the accuracy and the applicability of this method, some numerical results are given. We compare the solutions of the proposed method with the exact solutions for integer order case.

Keywords: Fractional Logistic Equation; Riemann-Liouville Fractional Integral; Caputo Fractional Derivative; Reproducing Kernel Hilbert Space.

1. INTRODUCTION

Logistic model was introduced to the population dynamics by Verhulst in 1838 [1] asa non-linear first order ordinary differential equation $\frac{dM}{dt} = \rho M\left(1 - \frac{M}{k}\right)$,where $M(t)$ is population at time t, $\rho > 0$ is Malthusian parameter, and k describes the carrying capacity.

Let $N(t) = \frac{M}{k}$, then the following standard logistic differential equation (LDE) results:

$$\frac{dN}{dt} = \rho N(1 - N). \qquad (1)$$

This equation has the known exact solution:$N(t) = \frac{N_0}{N_0 + (1 - N_0)e^{-\rho t}}$,where $N_0 = N(0)$ is related to the initial population.

Logistic differential equation has many applications, see [2-4]. Moreover, fractional calculus hasa great importance in describing some complex physical phenomena in many fields [5-11]. The fractional logistic differential equation (FLDE) has been obtained by replacing the first order derivative inEq. (1) by the fractional Caputo derivative D^{α}as

$$D^{\alpha}N(t) = \rho N(t)\big(1 - N(t)\big), \qquad t > 0, \quad \rho > 0, \ 0 < \alpha \leq 1, \qquad (2)$$

subject to the initial condition

$$N(0) = N_0, \ \ N_0 > 0. \qquad (3)$$

Most fractional differential equations don't have exact solutions. So, numerical methods are needed. Some of these techniques have been applied to solve FLDE [12-19]. In this paper, we use reproducing kernel Hilbert space method (RKHSM) to obtain numerical solution of Eq. (2). Reproducing kernel theory has important applications in mathematics, image processing, machine learning, finance and probability [20-24]. Hence a lot of research work has been devoted to the applications of RKHSM for wide classes of problems [25-31].

This paper is organized in five sections including the introduction. In section 2, some basics of fractional calculus and reproducing kernel theory are given. In section 3, a description of the RKHSM to solve the FLDE is discussed. In section 4, an example to show the reliability of the RKHSMis given. A brief conclusion is presented in section5.

2. PRELIMINARIES

In this section, we introduce some preliminaries of fractional calculus and reproducing kernel theory. For more details, see [29-31]. Throughout this paper $AC[a, b] = \{u: [a, b] \to \mathbb{R}: u$is absolutely continuous on$[a, b]\}$.

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Some basics of fractional calculus

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ over $[a,b]$ for a function g is $(J_{a+}^{\alpha} g)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g(z)}{(x-z)^{1-\alpha}}\, dz\,,\ x > a$. For $\alpha = 0$, J_{a+}^{α} is the identity operator.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha, 0 < \alpha < 1$ is defined by $(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt\,,\ x > a.$

Definition2.3. The Caputo fractional derivative of order α $(0 < \alpha < 1)$ is $({}^{C}D_{a+}^{\alpha} g)(x)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{g'(t)}{(x-t)^{\alpha}}\, dt.$

Theorem 2.4. Let $f(x) \in C[a,b]$ and $\alpha > 0$. Then $({}^{C}D_{a+}^{\alpha} J_{a+}^{\alpha} f)(x) = f(x)$.

Theorem 2.5. If $0 < \alpha \le 1$ and $f(x) \in AC[a,b]$, then $(J_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} f)(x) = f(x) - f(a)$.

Since the Caputo derivative has been used in this paper only with $a = t_0 = 0$, then the symbol D^{α} will be used instead of ${}^{C}D_{a+}^{\alpha}$.

2.2 Fundamental concepts of the reproducing kernel Hilbert space method

Definition 2.6. Let S be a nonempty abstract set. A function $K: S \times S \to \mathbb{C}$ is a reproducing kernel of the Hilbert space $\mathcal{H}$ if and only if

(1) $\forall t \in S,\ K(\cdot,) \in \mathcal{H}$,
(2) $\forall t \in S, \forall \varphi \in \mathcal{H},\ \big(\varphi(\cdot), K(\cdot,t)\big) = \varphi(t)$.

The function K is called the reproducing kernel function of $\mathcal{H}$ and a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS).

Definition 2.7. The space of functions $W_2^1[a,b]$ is defined as

$$W_2^1[a,b] = \{u: [a,b] \to \mathbb{R}: u \in AC[a,b], u' \in L_2[a,b]\}.$$

The inner product and the norm for $u, v \in W_2^1[a,b]$ are given by $\langle u, v \rangle_{W_2^1} = \int_a^b (u(t)v(t) + u'(t)v'(t))dt$ and $\|u\|_{W_2^1} = \sqrt{\langle u(t), u(t) \rangle_{W_2^1}}$, respectively.

Theorem 2.8. The space $W_2^1[a,b]$ is a complete RKHS with the reproducing kernel function $T_t(s)$ such that $T_t(s) = \frac{1}{2sinh(b-a)}[cosh(t+s-b-a) + cosh(|t-s| - b + a)]$.

Definition2.9. The space of real functions $W_2^2[a,b]$ is defined as follows:

$$W_2^2[a,b] = \{u: u, u' \in AC[a,b], u'' \in L_2[a,b], u(a) = 0\}.$$

The inner product and the norm for $u, v \in W_2^2[a,b]$ are given by $\langle u, v \rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(t)v''(t)dt$ and $\|u\|_{W_2^2} = \sqrt{\langle u(t), u(t) \rangle_{W_2^2}}$, respectively.

Theorem2.10. The space $W_2^2[a,b]$ is a RKHS and its reproducing kernel function $K_t(s)$ has the form $K_t(s) = \begin{cases} \frac{1}{6}(s-a)(2a^2 - s^2 + 3t(2+s) - a(6+3t+s)), & s \le t \\ \frac{1}{6}(t-a)(2a^2 - t^2 + 3s(2+t) - a(6+3s+t)), & s > t \end{cases}.$

3. THE RKHSM FOR SOLVING THE FLDE

Let us consider the FLDE in Eq. (2) with the initial condition Eq. (3). First we homogenize the initial conditionusing the substitution:$M(t) = N(t) - N_0$ to get$D^\alpha M(t) + D^\alpha N_0 = \rho(M(t) + N_0)(1 - M(t) - N_0)$. Since $D^\alpha N_0 = 0$, Eq. (2) and Eq. (3) become

$$D^\alpha M(t) = \rho(M(t) + N_0)(1 - M(t) - N_0), \qquad (5)$$
$$M(0) = 0.$$

Define the differential operator $L: W_2^2[a,b] \rightarrow W_2^1[a,b]$ such that $LM(t) = D^\alpha M(t)$.Hence, Eq. (5) can be rewritten as $LM(t) = \rho(M(t) + N_0)(1 - M(t) - N_0),\ t > 0.$

Now, to construct an orthogonal function system of the space $W_2^2[a,b]$, consider the dense set $\{t_i\}_{i=1}^{\infty}$of $[a,b]$, and let$\varphi_i(t) = T_{t_i}(t)$ and $\psi_i(t) = L^*\varphi_i(t)$, where L^* is the adjoint operator of L. In terms of the properties of the reproducing kernel $T_t(.)$, we obtain

$$\langle M(t), \psi_i(t)\rangle_{W_2^2} = \langle M(t), L^*\varphi_i(t)\rangle_{W_2^2} = \langle LM(t), \varphi_i(t)\rangle_{W_2^1} = LM(t_i), i = 1,2, \ldots.$$

Applying Gram-Schmidt orthogonalization process on $\{\psi_i(t)\}_{i=1}^{\infty}$produces the orthonormal function system $\{\overline{\psi_i}(t)\}_{i=1}^{\infty}$ of the space$W_2^2[a,b]$.Let $\overline{\psi_i}(t) = \sum_{l=1}^{i}\beta_{il}\psi_l(t)\,, i = 1,2,3, \ldots$where $\beta_{il}$are the orthogonalization coefficients, which are given by:

$\beta_{11} = \frac{1}{\|\psi_1\|_{W_2^2}}$,$\beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|_{W_2^2}^2 - \sum_{p=1}^{i-1}\langle\psi_i(t),\overline{\psi_p}(t)\rangle_{W_2^2}^2}}$,and$\beta_{il} = \frac{-\sum_{p=1}^{i-1}\langle\psi_i(t),\overline{\psi_p}(t)\rangle_{W_2^2}\beta_{pl}}{\sqrt{\|\psi_i\|_{W}^2 - \sum_{p=1}^{i-1}\langle\psi_i(t),\overline{\psi_p}(t)\rangle_{W_2^2}^2}}$, for i $> l$.

Theorem 3.1. If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of Eq. (5) is unique, then it has the form $M(t) = \rho\sum_{i=1}^{\infty}\sum_{l=1}^{i}\beta_{il}(M(t_l) + N_0)(1 - M(t_l) - N_0)\overline{\psi_i}(t)$.

The n-term approximate solution $M^n(t)$ of Eq. (5) is given by the finite sum such that

$$M^n(t) = \rho\sum_{i=1}^{n}\sum_{l=1}^{i}\beta_{il}(M(t_l) + N_0)(1 - M(t_l) - N_0)\overline{\psi_i}(t).$$

Hence, the approximate solution of Eq. (2) and Eq. (3) is$N^n(t) = M^n(t) + N_0$.

4. NUMERICAL EXAMPLE

A numerical example is included to demonstrate the efficiency of the RKHSM. Results obtained by this method for FLDE are compared with the exact solutionand are found in good agreement with each other.

Example 4.1. Consider the FLDE

$$D^\alpha N(t) = \frac{1}{2}N(t)\big(1 - N(t)\big), N(0) = \mu, \quad t > 0, 0 < \alpha \leq 1.$$

The approximate and exact solutions of different values of α are given in Table1and Figure1for $\mu = \frac{1}{4}$ and $\mu = \frac{1}{2}$. We take $n = 25$.

Table1: Numerical results for Example 4.1 for $t \in [0,1]$ using the RKHSM.

μ	t	Exact $N(t), \alpha = 1$	RKHS $N^n(t), \alpha = 1$	Absolute Error $\lvert N(t) - N^n(t)\rvert$	RKHS Solution $N^n(t)$	
					$\alpha = 0.9$	$\alpha = 0.7$
	0.2	0.26921	0.26921	9.9065×10^{-7}	0.27366	0.28539
	0.4	0.28934	0.28933	2.0637×10^{-6}	0.29536	0.30942

$\frac{1}{4}$	0.6	0.31032	0.31032	3.0952×10^{-6}	0.31692	0.33101
	0.8	0.33212	0.33212	4.0740×10^{-6}	0.33860	0.35127
	1.0	0.35466	0.35466	4.9951×10^{-6}	0.36046	0.37063
$\frac{1}{2}$	0.2	0.52498	0.52498	4.6269×10^{-7}	0.53049	0.54441
	0.4	0.54983	0.54983	9.2369×10^{-7}	0.55670	0.57167
	0.6	0.57444	0.57444	1.4050×10^{-6}	0.58122	0.59442
	0.8	0.59869	0.59869	1.9257×10^{-6}	0.60450	0.61446
	1.0	0.62246	0.62246	2.4991×10^{-6}	0.62668	0.63250

CONCLUSION

In this work, we applied the RKHSM to obtain approximate solutions for the non-linear FLDE. The fractional derivative was described in the Caputo sense. An example are given to show the efficiency of the proposed method. By comparing our results with the exact solution for integer order derivative, we observe that the proposed method yields accurate approximations. To see the effects of the fractional derivative on the logistic curve, we solved the same FLDE for different values of the fractional order. All computations have been performed using the Mathematica software package.

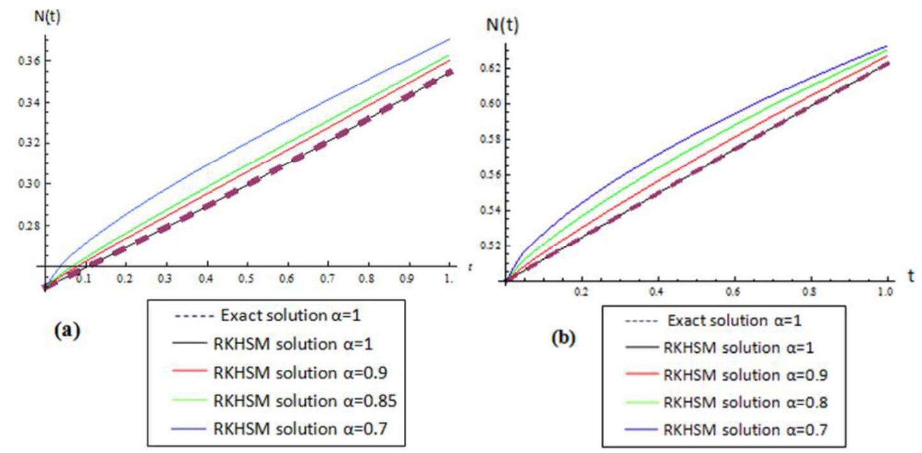

Figure 1: Graphical results for Example 4.1 with(a) $\mu = \frac{1}{4}$ and (b) $\mu = \frac{1}{2}$.

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AN APPLICATION OF TAYLOR SERIES METHOD IN HIGHER DIMENSIONAL FRACTAL SPACES

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ABSTRACT

The aim of this work is to extend the Taylor series method to higher dimensional fractal spaces. An analytical solution of higher dimensional fractional differential equations is provided with different fractal-memory indices in time and space coordinates simultaneously. To show the effectiveness of the proposed method, the method has been applied to three presented models in fractal 2D and 3D spaces. The attained closed-form series solutions are in a high agreement with the exact solutions for the corresponding equations when they projected into the integer space.

Keywords: Fractional partial differential equations; Taylor series method; Memory index

1. INTRODUCTION

Fractional calculus was appeared in 1695, in Leibniz letter to L'Hopital, definitely after the classical calculus was constructed. The evolution of the fractional calculus is due to the achievements of many mathematicians such as Liouville, Riemann, Abel, and many others, where the huge importance of the fractional calculus in sciences encouraged them. Many Phenomena such that, viscoelasticity, heat diffusion, mathematical biology, electrochemistry [13,7], are presented as fractional partial differential models, from this point arises the importance to solve these Models. With the result that, many mathematical integer-order methods have been generalized to fractional type to convoy the developments in mathematical sciences, such as residual power series method by Alquran et al. [3], and Abu-Arqub et al [17], differential transform method by Jaradat et al. [12] and Taylor series.

Taylor series has been generalized by many researchers throughout the ages, Riemann, Watanabe, Trujillo and many others [14]. But all of them ignore the power law memory of time fractional variable and treat only the space fractional variable or vice versa [12]. Whereas many recent studies show that the importance of combining the space variables to fractional scope. From this point appeared the most powerful generalization of Taylor series over time and space fractal spaces by Jaradat et al.[10,2]. These new expansions enable the researchers to solve fractional partial differential equations (FPDEs) in higher dimensional fractal spaces where the space and time coordiates are endowed with fractional derivatives ordering.

Several definitions for fractional derivative and integration were introduced, the most useful fractional derivative operator is Caputo definition which we adopt in our work with the following representation [9]:

$$\mathcal{D}_t^{\alpha}\left[u(\bar{x},t)\right]=\frac{\partial^{\alpha}u(\bar{x},t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(\bar{x},\kappa)}{\partial\kappa}\frac{d\kappa}{(t-\kappa)^{\alpha}}.$$

(1)

Where $\alpha\in$

2. THE CONSTRUCTION OF TAYLOR SERIES SOLUTIONS IN HIGHER DIMENSIONAL FRACTAL SPACES

In this section, we introduce two different solution formulas for the (2+1)-D and (3+1)-D FDEs that are presented into fractal 2D and 3D spaces. In some sense, the hybrid fractional Taylor's formulas in 2D and 3D fractal spaces are obtained. We should mention that these expansions were used before to solve different FPDEs into different dimensions [1,2,4,5,8,9,11,16].

Definition 2.1. An $(\alpha,\beta)-$fractional power series of the (2+1)-D FDEs in the fractal 2D space [9]:

$$\sum_{\substack{i+j=0\\ i,j\in\mathbb{N}}}^{\infty} g_{ij}(y)t^{i\alpha}x^{j\beta} = \underbrace{g_{00}(y)}_{i+j=0} + \underbrace{g_{10}(y)t^{\alpha} + g_{01}(y)x^{\beta}}_{i+j=1} + \cdots + \underbrace{\sum_{k=0}^{n} g_{n-k,k}(y)t^{(n-k)\alpha}x^{k\beta}}_{i+j=n} + \cdots \tag{2}$$

where g_{ij} are the coefficients of the series with function type.

The next Lemma and Remarkpresent the Taylor's formula in fractal 2D space, the proof of the lemma is similar to the proof of Lemma (2.2) in [9]:

Lemma 2.2. [9] *Let* $\upsilon(x,y,t)$ *has a FPS representation as* Eq. (2) *for* $(x,y,t)\in[0,R_x)\times I\times[0,R_t)$. *If* $\mathcal{D}_t^{r\alpha}\mathcal{D}_x^{s\beta}[\upsilon(x,y,t)]\in\mathcal{C}((0,R_x)\times I\times(0,R_t))$ *for* $r,s\in\mathbb{N}$, *then*

$$\mathcal{D}_t^{r\alpha}\mathcal{D}_x^{s\beta}[\upsilon(x,y,t)] = \sum_{i+j=0}^{\infty} g_{i+r,j+s}(y)\frac{\Gamma((i+r)\alpha+1)\Gamma((j+s)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta}. \tag{3}$$

Remark 1. [9]*By letting* $(x,t)=(0,0)$ *in*Eq.(3), *we have the following fractional form of Taylor's formula that is related to* Eq.(2)

$$\upsilon(x,y,t) = \sum_{i+j=0}^{\infty} \frac{\mathcal{D}_t^{r\alpha}\mathcal{D}_x^{s\beta}[\upsilon(x,y,t)]|_{(x,t)=(0,0)}}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta}. \tag{4}$$

In the case of converting the (2+1)-D FPDEs into the 3D fractal space, we replace the coefficients with function type by constant coefficients with the following formula:

$$\sum_{\substack{i+j+k=0\\ i,j,k\in\mathbb{N}}}^{\infty} a_{ijk}t^{i\alpha}x^{j\beta}y^{k\gamma} = \underbrace{a_{000}}_{i+j+k=0} + \underbrace{a_{100}t^{\alpha} + a_{010}x^{\beta} + a_{001}y^{\gamma}}_{i+j+k=1} + \cdots + \underbrace{\sum_{r=0}^{n}\sum_{s=0}^{r} a_{n-r,r-s,s}t^{(n-r)\alpha}x^{(r-s)\beta}y^{s\gamma}}_{i+j+k=n} + \cdots. \tag{5}$$

Remark 2.*Formulas Eq. (2) and Eq. (5) can be naturally extended to higher dimensional by adapting the coefficients.*

3. APPLICATIONS

Our purpose in this section is to present an analytical closed-form solution in fractal type for the considered models that are embedded into fractal 2D and 3D spaces. The solutions are found by using a parallel structure to the power series method with utilizing the previous representations (2), (5), and there extensions.

3.1.*Solution of Schrödinger mode in fractal 2D space*

Example 3.1.1. Consider the following (2+1)-D schrödinger initial value problem into the 2D fractal space [12]:

$$i\,\mathcal{D}_t^{\alpha}\big[u(x,y,t)\big]=\mathcal{D}_x^{2\beta}\big[u(x,y,t)\big]+u_{yy}(x,y,t), \qquad (6)$$

Subject to the initial condition

$$u(x,y,0)=\sin_{\beta}(x^{\beta})+\sin(y), \qquad (7)$$

where $\sin_{\beta}(x^{\beta})=\sum_{j=0}^{\infty}\frac{(-1)^j x^{(2j+1)\beta}}{\Gamma((2j+1)\beta+1)}$ is the fractional generalization of the function $\sin(x)$.

By substituting all the relevant quantitiesEq. (3) into Eq. (6) and Eq. (7), and equating the coefficients of like monomials from both sides, we get the following recursive equation:

$$i\,\frac{\Gamma\big((i+1)\alpha+1\big)}{\Gamma\big(i\alpha+1\big)}g_{i+1,j}(y)-\frac{\Gamma\big((j+2)\beta+1\big)}{\Gamma\big(j\beta+1\big)}g_{i,j+2}(y)-g_{ij}''(y)=0, \qquad (8)$$

with initial coefficients

$$g_{0,2j+1}(y)=\frac{(-1)^j}{\Gamma((2j+1)\beta+1)},\ g_{0,0}(y)=\sin(y). \qquad (9)$$

By solving the equation Eq. (8) recursively we get the following general coefficients:

$$g_{i,2j+1}(y)=\frac{(-1)^j (i)^i}{\Gamma((2j+1)\beta+1)\Gamma(i\alpha+1)},\ g_{i,0}(y)=\sin(y). \qquad (10)$$

So, the exact solution of the equation Eq. (6) is given with the following series solution form:

$$\begin{aligned}u(x,y,t)&=\sum_{i+j=0}^{\infty}\frac{(i)^i(-1)^j}{\Gamma(i\alpha+1)\Gamma((2j+1)\beta+1)}t^{i\alpha}x^{(2j+1)\beta}+\sin(y)\sum_{i=0}^{\infty}\frac{(i)^i}{\Gamma(i\alpha+1)}t^{i\alpha}\\&=\sum_{i=0}^{\infty}\frac{(i)^i t^{i\alpha}}{\Gamma(i\alpha+1)}\Big[\sum_{j=0}^{\infty}\frac{x^{(2j+1)\beta}}{\Gamma((2j+1)\beta+1)}+\sin(y)\Big],\\&=E_{\alpha}(it^{\alpha})[\sin_{\beta}(x^{\beta})+\sin(y)].\end{aligned} \qquad (11)$$

In particular, as the fractional derivative ordering $\alpha,\beta\to 1$ the solutionEq. (11) becomes $u(x,y,t)=e^{it}[\sin(x)+\sin(y)]$ which is the exact solution for the projection of Eq. (6) and Eq. (7) into the integer space.

3.2.*Solution of Schrödinger model in fractal 3D space*

Example 3.2.1. Consider the following (2+1)-D schrödinger initial value problem into the 3D fractal space:

$$i\,\mathcal{D}_t^{\alpha}\big[\upsilon(x,y,t)\big]=\mathcal{D}_x^{2\beta}\big[\upsilon(x,y,t)\big]+\mathcal{D}_y^{2\gamma}\big[\upsilon(x,y,t)\big], \qquad (12)$$

Subject to the initial condition

$$\upsilon(x,y,0)=\sin_{\beta}(x^{\beta})+\sin_{\gamma}(y^{\gamma}), \tag{13}$$

By substituting all the relevant quantities Eq. (5) into Eq. (12) and (31), and equating the coefficients of like monomials from both sides, we get the following recursive equation:

$$i\,\frac{\Gamma\big((i+1)\alpha+1\big)}{\Gamma\big(i\alpha+1\big)}a_{i+1,j,k}-\frac{\Gamma\big((j+2)\beta+1\big)}{\Gamma\big(j\beta+1\big)}a_{i,j+2,k}-\frac{\Gamma\big((k+2)\gamma+1\big)}{\Gamma\big(k\gamma+1\big)}a_{i,j,k+2}=0, \tag{14}$$

with initial coefficients

$$a_{0,2j+1,0}=\frac{(-1)^{j}}{\Gamma((2j+1)\beta+1)},$$

$$a_{0,0,2k+1}=\frac{(-1)^{k}}{\Gamma((2k+1)\gamma+1)}. \tag{15}$$

By solving the equation Eq. (13) recursively we get the following general coefficients:

$$a_{i,2j+1,0}=\frac{(-1)^{j}(i)^{i}}{\Gamma((2j+1)\beta+1)\Gamma(i\alpha+1)},$$

$$a_{i,0,2k+1}=\frac{(-1)^{k}(i)^{i}}{\Gamma((2k+1)\gamma+1)\Gamma(i\alpha+1)}. \tag{16}$$

So, the exact solution of the equation Eq. (12) is given with the following series solution form

$$\upsilon(x,y,t)=\sum_{i+j=0}^{\infty}\frac{(i)^{i}(-1)^{j}}{\Gamma(i\alpha+1)\Gamma((2j+1)\beta+1)}t^{i\alpha}x^{(2j+1)\beta}+\sum_{i+k=0}^{\infty}\frac{(i)^{i}(-1)^{k}}{\Gamma(i\alpha+1)\Gamma((2k+1)\gamma+1)}t^{i\alpha}y^{(2k+1)\gamma}$$

$$=\sum_{i=0}^{\infty}\frac{(i)^{i}t^{i\alpha}}{\Gamma(i\alpha+1)}\Big[\sum_{j=0}^{\infty}\frac{x^{(2j+1)\beta}}{\Gamma((2j+1)\beta+1)}+\sum_{k=0}^{\infty}\frac{y^{(2k+1)\gamma}}{\Gamma((2k+1)\gamma+1)}\Big]$$

$$=E_{\alpha}(it^{\alpha})[\sin_{\beta}(x^{\beta})+\sin_{\gamma}(y^{\gamma})] \tag{17}$$

In particular, as the fractional derivative ordering $\gamma\to 1$, the same fractal solutionEq. (12) is obtained, as $\alpha,\beta,\gamma\to 1$, the solutionEq. (16) becomes $\upsilon(x,y,t)=e^{it}[\sin(x)+\sin(y)]$ which is the exact solution for the projection of Eq. (12) and Eq. (13) into the integer space.

CONCLUSION

In this work, we present an analytical fractional solution of Schrödinger, Telegraph, and Heat-like models in higher dimensional fractal spaces. The solutions that obtained from the 3D hybrid Taylor series method show a high agreement with that obtained from the 2D hybrid Taylor series method as letting $\gamma\to 1$. Also by projecting the solutions that obtained from the $(\alpha,\beta,\gamma)-$FPS into the integer space, we perceive that the exact solutions of the integer-version of the proposed models are obtained.

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CHARACTERIZING WHEN THE POWERS OF A TREE ARE DIVISORS GRAPHS[4]

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ABSTRACT

A full characterization of when T^k with $k = 2, 3, 4$, is a divisor graph, was given by AbuHijleh et.al.. Moreover, same authors gave a characterization of T^k, when it is not a divisor graph, for any positive integer $k \geq 2$. In this paper, we give a full characterization of T^k, when it is a divisor graph with positive integer k greater than four.

Keywords: Tree; divisor graph; power of a graph.

1. INTRODUCTION

Throughout this paper a graph G means a finite simple graph, i.e. a graph without loops or multiple edges. A tree T is a connected graph that has no cycles. The distance between any two vertices x and y, is the length of a shortest path between them, denoted by $d(x, y)$. In a tree T, the path between two vertices is unique, hence the distance between two vertices is the number of edges in this path. An r-starlike tree T is represented by subdividing all edges of a star graph into paths (known by legs), where r is the number of legs. The diameter of a graph G, denoted by d or diam(G), is equal to sup$\{d(x, y): x, y \in V(G)\}$. The neighbour of a vertex u, denoted by $N(u)$, is the set of all vertices that are adjacent to u, then $|N(u)| = deg(u)$. A leaf vertex (end vertex), is a vertex u for which deg$(u)= 1$. The power graph G^k has the vertex set $V(G)$ and two vertices x and y are adjacent if and only if $d(x, y) \leq k$. For an oriented digraph D, a transmitter is a vertex having indegree 0, a receiver is a vertex having outdegree 0, while a vertex v is a transitive vertex if it has both positive outdegree and positive indegree such that $(u,w) \in E(D)$ whenever (u,v) and $(v,w) \in E(D)$. Whereas, if every vertex in a graph G is a transmitter, a receiver, or a transitive vertex, then D is a divisor orientation of G and G is a divisor graph. For an example, a complete graph and a bipartite graph (a tree is bipartite), see [6]. For undefined notions and terminology, the reader is referred to [4].

In 1983 Erdős et. al. [7] and Pollington [8], studied the length, $g(n)$, of a longest path in the divisor graph whose divisor labeling has range $\{1, 2, \ldots, n\}$. Since 1983, several papers appear about divisor graphs, such as [5] and [6]. A complete characterization of a divisor graph of powers of paths, cycles, hypercubes, folded hypercubes and caterpillars, beside T^2, were given in [1], [2] and [5]. Moreover, in 2015 AbuHijleh et. al. [3] gave a characterization of T^k, when it is not a divisor graph for any positive integer $k \geq 3$, beside T^3 and T^4, if they were a divisor graph. In this paper, we give a characterization of T^k, when it is a divisor graph for any positive integer k greater than four.

In the graph theory a divisor graphs also where studied under different names such as a comparability graph, a transitively orientable graph, a partially orderable graph, and a containment graph. Note that, every comparability graph is a perfect graph. A perfect graph is a graph in which the chromatic number of every induced subgraph is equal to size of largest clique of that subgraph. Whereas, perfect graphs are closely related to perfect channels in communication theory. Also, a novel application of a perfect graph relates to an urban science problem involving optimal routing of garbage trucks, see [9], and there are a lot of applications one can find it, especially for a power graph that have a main aspect in networking field.

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[*]Characterizing when the powers of a tree, are divisor graphs

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2. PRELIMINARIES

The following results give different characterizations of divisor graphs.

Theorem 2.1. *Let G be a graph, then G is a divisor graph if and only if G has a divisor orientation, see [6].*

Proposition 2.2. *Every induced subgraph of a divisor graph is a divisor graph, see [6].*

Theorem 2.3. *For any integer $k \geq 2$, if G is a graph of diameter $d \geq 2k+2$, then G^k is not a divisor graph, see [5].*

Theorem 2.4. *Suppose that T is a tree with diam(T) $\geq 2k - 2(l - 1)$, $1 \leq l \leq \left\lfloor \frac{k-1}{2} \right\rfloor$, $k \geq 3$ and T contains an induced subgraph that is isomorphic to $T_{k,l}$, see Figure 1. Then T^k is not a divisor graph, see [3].*

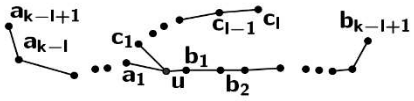

Figure 1: $T_{k,l}$.

Theorem 2.5. *Let T be a tree that is induced a 3-starlike tree T_e with length of each leg is $\frac{d_e}{2}$, where diam(T_e) = $d_e \geq 4$ (even), then T^{d_e-2} is not a divisor graph, see [1] and [3].*

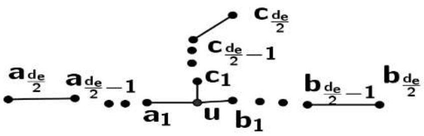

Figure 2: T_e.

According to Theorem 2.4, we have a specific form of *T* so that *T* is not induced an isomorphic subgraph of $T_{k,l}$ with odd positive integer *k*. The following definition gives a construction of an arbitrary tree *T*, so that $T_{k,l}$ is not induced in it.

Definition 2.6. First, construct a path, say P_d, with diam(P_d) = diam(*T*). Then label the consecutive vertices, after leaving $h+1$ vertices of P_d from one side, as follows $\{x_1, x_2, \ldots, x_m$: $m = d - 2h - 1$, $h = \frac{k-1}{2}\}$. Hence, there were $h + 1$ vertices in P_d after x_m, in the other side.

Second, construct subtrees on each interior vertex, without changes the diameter of *T*, and with a specific distance of x_i's, where each vertex has a specific name as given below.

Third, for each x_i's, consider the set of vertices $S_i = \{x_i,\ v_i = q_{l_j}^{x_i,h_i} : d(x_i, v_i) = h_i,\ i = 1, \ldots, m$ and $l_j = 1, \ldots,$ number of vertices in the level $h_i\}$. For any path *P* between v_i and x_j s.t. $i \neq j$, then $x_i \in P$. Moreover, at $i = 1, m$ we have $h_i = 1, \ldots, h$. At $i = 2, m-1$ we have $h_i = 1, \ldots, h-1$ and continuing by this manner, to reach to the middle. Also, define the sets, in the level $h+1$ of x_1 to be, $S_{1,i} = \{ v_{i,l_j} = q_{l_j}^{x_1,h+1,i}$: d(x_1, v_{i,l_j}) = $h + 1$ and $l_j = 1, \ldots,$ number of leaves in the level $h+1$, whereas for each $i = 1, \ldots, deg(x_1) - 1$, the path P_i from v_{i,l_j} to x_1 passes through $q_i^{x_1,1}$ }. Similarly define the sets, in the level $h+1$ of x_m to be $S_{m,i}$, where $i = 1, \ldots, deg(x_m) - 1$.

Fourth, if $d = 2k + 1$, we find that $S_{h+1} = \{x_{h+1}\}$ and $S_{h+2} = \{x_{h+2}\}$. At $d = 2k$ or less, we delete the set of vertices $S_{\lceil \frac{m}{2} \rceil}$ and adjacent $x_{\lceil \frac{m}{2} \rceil - 1}$ with $x_{\lceil \frac{m}{2} \rceil + 1}$. Then rename the vertices in $S_{\lceil \frac{m}{2} \rceil + 1}$ to be $S_{\lceil \frac{m}{2} \rceil}$, and similarly the successive sets till S_m, see examples in Figure 3 and Figure 4.

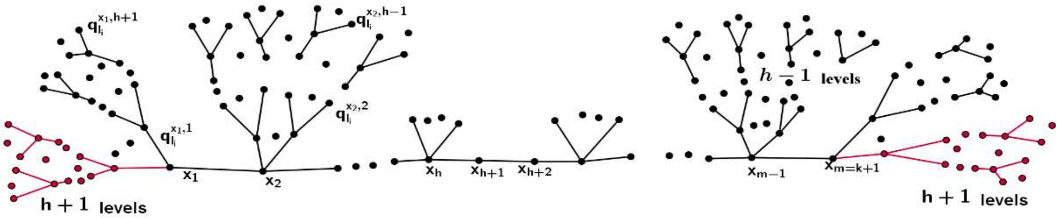

Figure 3: T with $d = 2k+1$.

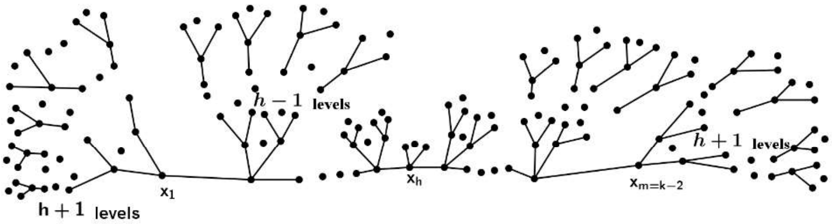

Figure 4: T with $d = 2k-2$.

Similarly, by using Theorem 2.3 and Theorem 2.4, we have a specific form of T, so that T is not induced an isomorphic subgraph of $T_{k,l}$ or T_e with even positive integer k.

3. CHARACTERIZING WHEN POWERS OF A TREE T^K ARE DIVISOR GRAPHS, FOR $k \geqslant 2$.

For $k = 3$, T^3 was characterized by AbuHijleh et. al. [3]. But if $k \geq 5$ with odd positive integer k, we give a characterization in the following theorem, which is a generalization of result at $k = 3$.

Theorem 3.1. *Suppose that T is a tree with diam(T)* $\leq 2k + 1$, $k = 2h + 1$ *and* $k \geq 5$. *Then* T^k *is a divisor graph if and only if T is not induced a subgraph that is isomorphic to* $T_{k,l}$.

Proof. Assume that T is induced a subgraph that is isomorphic to $T_{k,l}$. Then, by Theorem 2.4, T^k is not a divisor graph.

Conversely, assume that T with diam(T) $\leq 2k + 1$, is not induced a subgraph that is isomorphic to $T_{k,l}$, where $l = 1, \ldots, h$. Then T will take a certain form, that is given in Definition 2.6. Otherwise, if you add an edge to any leaf (without changes the diameter), you will get an induced subgraph that is isomorphic to $T_{k,l}$, see Figure 3 and Figure 4 as an examples.

Moreover, by using sets in Definition 2.6, there are three cases to consider w. r. to diameter of T.

Case 1: For $2k - 2 \leq diam(T) \leq 2k + 1$.

- $S_A = (\bigcup_{i=1}^{i=h} S_i) \cup (\bigcup_{i=1}^{i=j} S_{1,i})$ with $j = deg(x_1) - 1$.
- $S_B = (\bigcup_{i=h+1}^{i=m} S_i) \cup (\bigcup_{i=1}^{i=j} S_{m,i}) - \{x_r, x_t\}$ *with* $j = deg(x_m) - 1$.

Note that: (a.) Let $x_r = x_{h+1}$. (b.) Let $x_t = x_{h+2}$. (c.) At $d = 2k - 2$ and $k = 5$, we have $x_{h+1} = x_m$, so let $x_t = q_1^{x_m,1}$.

Case 2: For $2k - h \le diam(T) \le 2k - 3$.

– $S_A = S_{A1} \bigcup S_{A2}$, where $S_{A1} = (\bigcup_{i=1}^{i=\lceil \frac{m}{2} \rceil} S_i) \bigcup (\bigcup_{i=1}^{i=j} S_{1,i})$ with $j = deg(x_1) - 1$, and

$S_{A2} = (\bigcup_{i=\lceil \frac{m}{2} \rceil +1}^{i=h} S_i) -$ {u: $d(u,v) > k$ with $v \in S_{1,i's}$ }.

– $S_B = S_{B1} \bigcup S_{B2}$, where $S_{B1} = (\bigcup_{i=\lceil \frac{m}{2} \rceil +1}^{i=h} S_i) - S_{A2}$ and $S_{B2} = (\bigcup_{i=h+1}^{i=m} S_i) \bigcup (\bigcup_{i=1}^{i=j} S_{m,i})$

$-\{x_r, x_t\}$, with $j = deg(x_m) - 1$.

Note that: (a.) Let $x_r = x_{h+1}$. (b.) Let $x_t = x_{h+2}$, for $2k - h + 1 \le d \le 2k - 3$. (c.) At $d = 2k - h$, we have $x_{h+1} = x_m$, so let $x_t = q_1^{x_m,1}$. (d.) For $k = 7$, we have only one case is $d = 2k - h = 2k - 3$, hence consider case (c.) for it. (e.) For $k = 5$, we have $d = 2k - h = 2k - 2$ and that's in case 1.

Case 3: For $k + 2 \le diam(T) \le 2k - h - 1$.

– $S_A = S_{A1} \bigcup S_{A2}$, where $S_{A1} = (\bigcup_{i=1}^{i=\lceil \frac{m}{2} \rceil} S_i) \bigcup (\bigcup_{i=1}^{i=j} S_{1,i})$ with $j = deg(x_1) - 1$, and

$S_{A2} = (\bigcup_{i=\lceil \frac{m}{2} \rceil +1}^{i=m} S_i) - (\{$u: d(u,v) > k with v $\in S_{1,i's}\} \bigcup \{x_r\})$.

– $S_B = S_{B1} \bigcup S_{B2}$, where $S_{B1} = (\bigcup_{i=\lceil \frac{m}{2} \rceil +1}^{i=m} S_i) - (S_{A2} \bigcup \{x_r, x_t\})$ and $S_{B2} = \bigcup_{i=1}^{i=j} S_{m,i}$, with j

$= deg(x_m) - 1$.

Note that, in this case $m \le h$. Hence $x_r = q_1^{x_m, j_1}$ and $x_t = q_1^{x_m, j_1+1}$, where $x_t \in N(x_r)$ and $d(x_r, v) = k$ with $v \in S_{1,i's}$.

Let D be an orientation of T^k, where $E(D) = A \cup B \cup C$ and $A, B, \& C$ are defined as follows:

(1) For $A \subset E(D)$:
   (i) For $u \in S_A$, then $(u, x_r) \in A \subset E(D)$.
   (ii) For $u, v \in S_A$ and $d(x_r, u) > d(x_r, v)$, then $(u, v) \in A \subset E(D)$.
   (iii) For $u, v \in S_A$, $d(x_r, u) = d(x_r, v)$ and $d(u, v) \le k$. Let $(u, v) \in A \subset E(D)$.
   (iv) For $u, v \in S_A$ and $d(u,v) = k + 1$, then $u \in S_{1,i_1}$ and $v \in S_{1,i_2}$, where $i_1 \ne i_2$. Then $uv \notin E(T^k)$ and for any $z \in S_A$ different than u and v, we have two cases:
   If $d(u, z) \le k$ and $d(v, z) \le k$. Hence, $d(x_r, z) < d(x_r, u)$ and $d(x_r, z) < d(x_r, v)$, then $\{(u, z), (v,z)\} \subset A \subset E(D)$.
   If $d(u, z) \le k$ and $d(v, z) = k+1$. Hence, $u, z \in S_{1,i_1}$ and $v \in S_{1,i_2}$, where $i_1 \ne i_2$. Then $(u, z) \in A \subset E(D)$ and $zv \notin E(T^k)$.

(2) For $B \subset E(D)$:
   (i) For $v \in S_B$, then $(x_t, v) \in B \subset E(D)$.
   (ii) For $u, v \in S_B$ and $d(x_{\lfloor \frac{m}{2} \rfloor}, u) < d(x_{\lfloor \frac{m}{2} \rfloor}, v)$, then $(u, v) \in B \subset E(D)$.

(iii) For $u, v \in S_B$, $d(x_{\lfloor \frac{m}{2} \rfloor}, u) = d(x_{\lfloor \frac{m}{2} \rfloor}, v)$ and $d(u, v) \leq k$. Let $(u, v) \in B \subset E(D)$.

(iv) For $u, v \in S_B$ and $d(u, v) = k + 1$, then $u \in S_{m,i_1}$ and $v \in S_{m,i_2}$, where $i_1 \neq i_2$. Then $uv \notin E(T^k)$ and for any $z \in S_B$ different than u and v, we have two cases:

a. If $d(u,z) \leq k$ and $d(v,z) \leq k$. Hence $d(x_{\lfloor \frac{m}{2} \rfloor},z) < d(x_{\lfloor \frac{m}{2} \rfloor},u)$ and $d(x_{\lfloor \frac{m}{2} \rfloor}, z) < d(x_{\lfloor \frac{m}{2} \rfloor},v)$, then $\{(z, u), (z,v)\} \subset B \subset E(D)$.

b. If $d(u, z) \leq k$ and $d(v, z) = k+1$. Hence $u, z \in S_{m,i_1}$ and $v \in S_{m,i_2}$, where $i_1 \neq i_2$. Then $(u, z) \in B \subset E(D)$ and $zv \notin E(T^k)$.

(3) For $C \subset E(D)$:

(i) $(x_t, x_r) \in C \subset E(D)$.

(ii) For $u \in S_B$ and $d(x_r, u) \leq k$, then $(u, x_r) \in C \subset E(D)$.

(iii) For $u \in S_A$ and $d(x_t, u) \leq k$, then $(x_t, u) \in C \subset E(D)$.

(iv) For $u \in S_A$, $v \in S_B$ and $d(u, v) \leq k$, then $(v, u) \in C \subset E(D)$.

It is enough to show that every vertex of D is a transmitter, a receiver, or a transitive vertex.

(1) For diam(T) = $2k + 1$, we have x_r is a receiver. Also we have a set of receivers, say S_r, where for each $S_{m,I}$ with $i = 1, \ldots,$ deg$(x_m) - 1$, $S_{m,I}$ is induced a clique in T^k and we have only one receiver in each set of $S_{m,i}$, hence $|S_r| =$ deg$(x_m) - 1$. But for diam(T) $\leq 2k$, we have only one receiver is x_r and $S_r = \phi$.

(2) For a transmitter vertex we have x_t. Also we have a set of transmitter, say S_t, where for each $S_{1,I}$ with $i = 1, \ldots,$ deg$(x_1) - 1$, $S_{1,I}$ is induces a clique in T^k and we have only one transmitter in each set of $S_{1,i}$, hence $|S_t| =$ deg$(x_1) - 1$.

(3) For a transitive vertex, we have three cases to consider:

(i) Let $u, v, z \in S_A - S_t$ and $\{(u, v), (v, z)\} \subset A \subset E(D)$. Then $d(u, x_r) \geq d(v, x_r) \geq d(z, x_r)$, which implies that $d(u,z) \leq k$ and $(u, z) \in A \subset E(D)$.

(ii) Let $u, v, z \in S_B - S_r$ and $\{(u, v), (v, z)\} \subset B \subset E(D)$. Then $d(u, x_{\lfloor \frac{m}{2} \rfloor}) \leq d(v, x_{\lfloor \frac{m}{2} \rfloor}) \leq d(z, x_{\lfloor \frac{m}{2} \rfloor})$, which implies that $d(u,z) \leq k$ and $(u, z) \in B \subset E(D)$.

(iii) Let $u, v \in S_A - S_t$ and $w, z \in S_B - S_r$:

If $(z, u) \in C \subset E(D)$ and $(u, v) \in A \subset E(D)$, then $d(u, x_r) \geq d(v, x_r)$. Which implies $d(v,z) \leq d(u,z) \leq k$, hence $(z, v) \in C \subset E(D)$.

If $(z, w) \in B \subset E(D)$ and $(w, u) \in C \subset E(D)$, then $d(z, x_{\lfloor \frac{m}{2} \rfloor}) \leq d(w, x_{\lfloor \frac{m}{2} \rfloor})$. Which implies $d(u, z) \leq d(u, w) \leq k$, hence $(z, u) \in C \subset E(D)$.

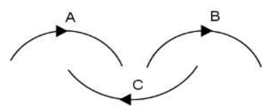

Figure 7: The sketch of the direction in D.

The sketch of the direction in D is represented in Figure 7. Thus, D is a divisor orientation of T^k. Hence by Theorem 2.1, T^k is a divisor graph for $k+2 \leq diam(T) \leq 2k+1$. For T with diam$(T) \leq k+1$, T^k is an induced subgraph of T^k with diam$(T) = k+2$. So that, by above work and Proposition 2.2, T^k is a divisor graph.□

For $k = 2, 4$, T^k was characterized by AbuHijleh et. al. in [1] and [3], respectively. But if $k \geq 6$ with even positive integer k, then the following theorem characterizes it, where it is a generalization of result at $k = 2, 4$.

Theorem 3.2. *Suppose that T is a tree with diam(T) ≤ 2k + 1, k = 2h and k ≥6. Then* T^k *is a divisor graph if and only if T has no induced subgraph that is isomorphic to* $T_{k,l}$ *or a subgraph that is isomorphic to* T_e.

Note that, the proof of Theorem 3.2 is like one in Theorem 3.1 with minor differences. Finally, by Theorem 3.1, Theorem 3.2 and results in AbuHijleh [1] and [3], we give a full characterization, for when T^k is a divisor graph with positive integer $k \geq 2$.

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